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Abstract

This paper is written as a tribute to Professors Robert Merton and Myron Scholes, winners of the 1997 Nobel Prize in economics, as well as to their collaborator, the late Professor Fischer Black. We first provide a brief and very selective review of their seminal work in contingent claims pricing. We then provide an overview of some of the recent research on stock price dynamics as it relates to contingent claim pricing. The continuing intensity of this research, some 25 years after the publication of the original Black-Scholes paper, must surely be regarded as the ultimate tribute to their work. We discuss jump-diffusion and stochastic volatility models, subordinated models, fractal models, and generalized binomial tree models, for stock price dynamics and option pricing. We also briefly address questions as to whether derivatives trading poses a systemic risk in the context of models in which stock price movements are endogenized.

1. Introduction

In October 1997, Robert Merton and Myron Scholes were awarded the Nobel Prize in economics for their pioneering work, with the late Fischer Black, on contingent claims pricing and their now-famous option pricing formula. The Nobel prize, along with the fact that the original 1973 Black-Scholes paper is still one of the most heavily cited in finance[1], presumably establishes the *academic* merit of their work as strongly as such merit can ever be established. But in addition, the Nobel committee noted, "...thousands of traders and investors use the formula every day." As the *Economist* panned: "Economists may sometimes seem about as useful as a chocolate tea-pot, but as this year's Nobel prize for economics shows it isn't always so."

This paper is intended as a tribute to the work of Professors Merton and Scholes, along with Professor Black. The preceding testimony speaks to the seminal nature and widespread influence of their work that will be especially well known to the readers of this journal. So we only briefly review it here, and concentrate instead on the vigorous research that their work is stimulating to this day. The intensity of this research, more than twenty years after the publication of their original articles, is surely the ultimate tribute to Professors Black, Merton, and Scholes.

2. The Original Ideas

Options are not themselves novel securities. Joseph de la Vega, in his 1688 account of the operation of the Amsterdam Exchange, provides a description of the "opsies" traded there, which bore a striking similarity to modern call and put options. Further back, the ancient Greeks apparently knew the basic principles behind the use of options.[2] In fact, given our general intuitive understanding of the value of "having options" in everyday life[3], it is undoubtedly true that work on options could be traced back indefinitely. Here, we start with Bachelier's quite amazing work that applied Brownian motion concepts to

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[1] Alexander and Mabry (1994). The citation counts presumably don't include a reference simply to "Black-Scholes", which is itself testimony to the academic recognition of their work.

[2] See, for example, Bernstein's (1992) reference to Aristotle's story in Book I of *Politics* describing a philosopher who foresaw a good harvest of olives and purchased call options on olive presses that would be in big demand if the harvest were large.

[3] "Option" is derived from the Latin word *optio*, which means choice.

stock market fluctuations some five years before Einstein's classic 1905 paper, and provided a rigorous theoretical modeling of options. Bachelier's work lay dormant until it was discovered by Paul Samuelson and his young co-workers, Merton and Scholes prominent among them, when they became interested in the option-pricing problem. Bachelier constructed a mathematical model of a Brownian motion, which he used to analyze the theoretical value of options on bond futures. Notwithstanding that Bachelier's model was based on normal rather than lognormal diffusion processes, and that he used an equilibrium demand-supply analysis with risk-neutral investors (rather an arbitrage/relative-value analysis), he presented a fairly complete treatment of Brownian motion and an option pricing formula.

The payoff on a (European) call option on a stock depends upon the extent and likelihood of the stock's price exceeding the option's exercise value when the option matures. It might thus seem that it is necessary to make an assumption about expected increases in the stock's price in order to value a call option. The critical insight that Professors Black, Merton, and Scholes had back at MIT in the early 1970s was that such reasoning is incorrect. Rather, if the market's projections about the stock's future prospects are already reflected in the stock's current price, the option can be valued ***relative to the observable current stock price.*** The valuation uses a transformed probability distribution for the stock price, and the expectation of this transformed distribution can conveniently be set equal to the riskless rate of interest. The option can be so valued relative to the stock price because its payoffs can be dynamically replicated by a levered position in the stock. Moreover, since the option's payoffs can be replicated by the levered position in the stock, the formula valuing the option relative to the stock holds so long as arbitrage opportunities can be ruled out. Thus, at least in the "plain vanilla" case, the options valuation formula is more robust than would be an equilibrium valuation model in which investors' net demand schedules have to be specified.

The arbitrage-free pricing approach not only produced a pricing formula for options. Because it relied upon the replication of the options' payoffs, it *ipso facto* demonstrated how the options being priced could actually be ***created*** or "synthesized". The payoff on the option at some future maturity date is replicated by a ***dynamic*** rule for trading in the underlying stock over the option's life. A call option on a stock, for example, is replicated by buying more stocks when the price goes up, and reducing the stock position when the price goes down.

This idea of dynamically replicating option payoffs extended the basic concept of static replication. Static replication is basic, *e.g.* a contract to exchange yen for dollars can be replicated by a portfolio containing a contract to exchange yen for DM and a second contract to exchange DM for dollars. The focus on dynamic replication of option payoffs led in turn to a more general economic insight. A strategy for trading securities that pay off in just a few states *each period* will enable an investor to replicate desired payoffs on many possible states ("scenarios") far into the future. The ability to transact through time is a substitute for holding a large number of long-term securities each maturing in specific states and at specific points in the future. This tradeoff will be particularly attractive if it is cheaper to trade a small number of securities over time than it is to maintain markets in a large number of buy-and-hold long-term securities that are state-dependent. In a competitive securities market, we would expect to find that customized derivative securities ("structured products") created by specialist financial firms on an *ad hoc* basis will eventually be priced at those firms' costs of hedging the securities, which are presumably below the purchasers' costs. For example, a double lookback option whose state-dependent payoffs depend upon the maximum and/or minimum prices of one or two assets over a given period can be written by a financial firm and then hedged by dynamically trading the one or two assets up to the option's maturity. An interesting question concerns which derivatives contracts eventually become "commoditized" as standardized contracts listed on Exchanges, and which remain OTC contracts to be written and hedged by specialists trading in the standardized securities on which the contracts are written.

The dynamic hedging and pricing of customized OTC claims is often explained using the "tree" diagram introduced by Cox, Rubinstein, and Ross (1979).[4] Indeed, these tree diagrams have brought options understanding "to the masses" in the same way that Feynman diagrams made quantum electrodynamics (QED) accessible to a wide audience in physics. When the dynamics for the stock price (or other variable on which an option's value depends) are represented in a tree, it is well known that the option can be valued by backward recursion through the tree. American options can be handled readily. The tree representation for a stock's price dynamics, and hence for the payoffs on an option on the stock, is similar to the explicit finite difference method for numerically solving the differential equations originated

[4] Another Nobel laureate, William Sharpe, is credited with conceiving the tree diagrams during the mid-day break at a conference near the Dead Sea in Israel!

by Black, Merton, and Scholes (see Geske and Shastri (1985) for a comparison of tree and finite difference methods).

In the 1990s, a good deal of research has involved the calibration of tree models for the dynamics of the underlying variables using observed prices of options themselves. For example, given a set of identification assumptions, Rubinstein (1994) infers the binomial tree and associated terminal probability distribution of the stock market index from a cross-section of index options with the same maturity. Likewise, Ho and Lee (1986) proposed calibrating fixed income models with the observed prices of contingent claims and bonds, something which is now done widely in practice. Black and Scholes anticipated, and perhaps stimulated, this research when they pointed out that options prices implied a value for the volatility parameter in their model. At the same time, there is an interesting and still-open issue raised in these implied volatility calculations, inference of the parameters of binomial trees, etc. Typically, the tree (or more generally, the parameters of the conditional probability distribution for the underlying variable) is re-estimated daily. For example, in the straightforward case of the Black-Scholes model, an implied volatility is calculated today by plugging in today's option price (for, say, the short-maturity at-the-money option) into the Black-Scholes model which is based on the assumption that the volatility is constant. But we are not taking this assumption seriously if we know that, tomorrow, we will repeat the exercise and calculate a new implied volatility, quite likely different from today's. The type of inconsistency is essentially the same as "time inconsistency" in the more general economics literature.

3. Generalizing the Stock Price Distribution in Contingent Claim Pricing

The underlying assumption of the Black-Scholes-Merton arbitrage-free pricing of stock options is that an option's payoff can be replicated by a levered position in the stock that is dynamically adjusted over time. As is now well known, this requires that stock prices have a continuous sample path, and that the volatility of the stock price be non-stochastic. The Black-Scholes formula was derived for the special case where the volatility was not only non-stochastic, but also constant, *i.e.* stock prices follow a geometric Brownian motion.[5]

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[5] Here, we use the terms lognormality and geometric Brownian motion interchangeably. In the geometric Brownian motion model for stock prices, the distribution of the stock price at time $T>t$ conditional on the

Samuelson (1965) and other earlier writers suggested that speculative prices should have a martingale property in competitive securities markets. Geometric Brownian motion is just a special martingale where the conditional probability distribution of (relative) stock price increments is assumed constant, an assumption that makes it particularly attractive because of the tractability it confers. However, the martingale property of prices doesn't rule out conditional probability distributions for price changes that vary over time, and it certainly doesn't preclude serially dependent variation in stock price volatility. Discontinuities and other non-Gaussian behavior of stock prices can be consistent with competitive markets.

In reality, it is widely agreed today that observed distributions of changes in log stock prices and log exchange rates almost always have "fat tails," *i.e.* they have excess kurtosis relative to the Gaussian, particularly over short intervals such as a day or less. Related to the fat tails is evidence that the volatility of exchange rates and stock prices changes stochastically over time; and that those changes are persistent, *i.e.* periods of high or low volatility tend to cluster together. The persistence of variation in stock prices and exchange rates seems to have a long memory, *e.g.* Taylor (1986) and Ding, Granger, and Engle (1993). The latter document positive autocorrelations in the absolute value of S&P 500 index returns up to 2,700 days. Most of the long memory in S&P Index returns seems to be due to pre-World War II realizations, but similar results have been found for the Nikkei index and for individual stock returns.

A more recent but burgeoning literature analyzing option price themselves also suggests that options don't seem to be priced as if the underlying distribution of stock prices and exchange rates is geometric Brownian motion. There are smiles and skews in implied volatilities inferred from the observed pricing of stock and currency options when the implied volatilities are calculated using the Black-Scholes model, at least since late 1987. For currency options, the implied volatilities tend to be higher for in-the-money and out-of-the-money options than for at-the-money options. For stock index options, the implied volatility tends to be higher for out-of-the-money options, with out-of-the-money puts trading at higher implied volatilities than out-of-the-money calls. These "smiles" in implied volatilities could indicate that the market is pricing perceived kurtosis and possibly skewness in the prices of underlying stocks.

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stock price at time t is a lognormal random variable. Mandelbrot (1997, p. 255) criticizes the general use of "lognormal" model as a synonym for "geometric Brownian" model.

Rather than using the cross-sections of option prices to simply the calculation of an implied volatility parameter for an assumed Gaussian distribution of underlying stock prices, Derman and Kani (1994), Dupire (1993)(1994), Rubinstein (1994), and Shimko (1993) proposed inferring the entire (risk-neutral) terminal probability distribution for a stock index implied by the prices of options with different strikes, along with a binomial tree for stock price dynamics that is consistent with that terminal distribution. Generally, these authors find that options prices imply more left-skewness implied by stock index option prices after October 1987. Kuwahara and Marsh (1994) used similar procedures to infer the implied distribution of individual Japanese equities on which equity warrants are traded. These warrants are long-maturity options, and under certain simplifying assumptions, Kuwahara and Marsh found that, if anything, the Japanese equity warrants tend to have implied distributions which are opposite that for stock market indexes, *i.e.* "frowns" instead of smiles.[6] Implied volatilities for shorter-term options on individual stocks also seem to be roughly "flat" with respect to in-the-moneyness of the options.

It is very interesting that stock indexes and individual stocks would have different implied distributions and volatility smiles since, after all, the index is just an aggregate of the stocks. One interpretation is that the two results together tell us something about the behavior of correlations among stock returns, *e.g.* that the *correlations* increase when there are big negative changes across the prices of all stocks in the index. In two other respects, the difference between index and individual stocks may simply not be real. First, the deep in-the-money and deep out-of-the-money options are often quite illiquid and hence the volatilities they imply are unreliable; at the same time, if one infers the slope of the smile for closer-to-the-money options, it tends to look closer to the index but misses the most informative points. A second interpretation for the Japanese equity warrant results is that *a term structure* of implied volatilities is being confounded with an equal-maturity cross-section of implied volatilities because it is hard to equalize maturities across the equity warrants --- we know that, at least for indexes, implied volatilities often display a significant term structure with respect to option maturity; though the slope of this term structure seems to shift from positive to negative across time and across countries.

[6] This in spite of the stylized fact that implied volatilities across all warrants tend to be higher for out-of-the-money warrants.

In summary, there now seems to be general agreement among researchers, both those looking directly at the stock returns data and those looking at options prices, that stock price behavior deviates systematically from the geometric Brownian motion model underlying the famous Black-Scholes formula. Much of the research extending the Black-Scholes-Merton work is then focused on extending the geometric Brownian motion model[7]. In the remainder of this section, we review the extended statistical models of stock price behavior. In Section 5, we briefly discuss structural models in which stock price dynamics are affected by feedback --- many practitioners and academics alike feel that these feedback affects are particularly important to understanding price behavior when markets are "under stress."

The statistical models are grouped into four related categories: First are the jump-diffusion and stochastic volatility models, in Section 3.1. Second are the subordinated models for conditional stock prices in which the "clock time" scale on which prices are observed is a deformation of an operational or trading time scale on which prices evolve. Third, we discuss Benoit Mandelbrot's multifractal model for stock prices in which stock price moments are restricted to satisfy certain scaling relationships as the time-scale of observation changes. Fourth are generalizations of implied binomial tree procedures which involve calibrating stock price dynamics (under certain identification assumptions) with option prices, rather than estimating the stock price models from the stock price data.

3.1 Jump-Diffusion and Stochastic Volatility Models

One approach to incorporating kurtosis and skewness in option valuation is to characterize the stock price distributions or approximations which would do a good job of capturing the observed kurtosis and skewness, and then to value options for these stock price distributions. For example, Robert Merton examined some twenty years ago the impact of jumps in stock prices, and the resulting fat tails in their probability distribution, on option values..[8] Moreover, in conversation, Merton would often point out how options traders would appear to make profits for many years by selling apparently overpriced options, only to see those profits (and perhaps more) wiped out in just one trading session when the overlooked jumps

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[7] Other research, not reviewed here, looks at how transactions costs, which prevent the perfect replication of contingent claim payoffs, affect contingent claim valuation.

[8] Published in Merton (1976).

occurred. After these trading sessions, the traders would be carried out on stretchers! (having had heart attacks).

It is well known that jump-diffusion models and stochastic volatility models[9] can be represented as countable mixtures of normals, and that mixtures of normal distributions have fat tails relative to a normal density; see Merton (1976) and Clark (1973), Blattberg and Gonedes (1974) respectively). However, though jump-diffusion models can generate considerable kurtosis in probability distributions of short-run returns, the kurtosis tends to "die out" for returns computed over longer intervals. This decay is also a feature of early models in which stock returns are subordinated to an independent draw from a given distribution for volatility each period; for example, if variance is drawn independently from an inverted gamma distribution each period and returns conditional on the realized volatility are normal, then the subordinated distribution is Student-*t* with "short run" fat tails). As discussed below, Mandelbrot (1997) is one of the sharpest critics of this lack of time scaling of stock return distributions.

But variation in stock return volatility does not appear to be independent from period to period. Rather, it clusters in time. One can easily convince oneself that there is likely to be at least first-order temporal dependence in volatilities by squaring daily returns and then running a simple first order autoregression. But as noted above, evidence by Taylor (1986) and Ding, Granger, and Engle (1993) suggests positive autocorrelations in the absolute value of S&P500 index returns up to 2,700 lags.

Probably the most well-known and easily implementable models encompassing temporal dependence are the ARCH (Autoregressive Conditional Heteroscedasticity) and GARCH (Generalized Autoregressive Conditional Heteroscedasticity) models, and their progeny such as EGARCH, FARCH, PARCH, IGARCH, FIGARCH. The GARCH(p,q) model for the period-t conditional variance of a stock return σ_t^2 is $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$ where ε_{t-i}^2 is the squared shock to the return in period t-i. Typically when low-order p and q GARCH models are fit to the data, the ARCH and GARCH parameters are close to an Integrated GARCH (IGARCH) model. For example, for a typical fit of the GARCH(1,1) model (*i.e.* p=1, q=1) to the daily returns on Daiwa House stock, Kuwahara and Marsh (1992) found $\beta = 0.87$ and

[9] A good survey of stochastic volatility models can be found in Ghysels, Harvey, and Renault (1996).

$\alpha = 0.08$, close to an Integrated GARCH (IGARCH) model in which $\alpha + \beta = 1$. But even if $\alpha + \beta = 1$, the effect of a lagged squared return shock on forecasts of future volatility still dies out exponentially (*e.g.* Ding and Granger (1996)). In this sense, neither the GARCH nor IGARCH models parsimoniously produce long-run persistence in volatilities.

A model which does, and thus is probably considered the "state of the art" ARCH/GARCH model for volatility, is the Fractionally Integrated Exponential GARCH (FIEGARCH) model:

$$\log(\sigma_t^2) = \varpi + \phi(L)^{-1}(1-L)^{-d}[1-\lambda(L)]g(\xi_{t-1}) ,$$

where $g(\xi_t) = \theta\xi_t + \gamma[|\xi_t| - E|\xi_t|]$, the unexpected return in period t is defined as $\varepsilon_t = \xi_t\sigma_t$, and all the roots of $\phi(L)$ and $\lambda(L)$ lie outside the unit circle (see Bollerslev and Mikkelsen (1996)). Beside long memory, this model accounts for the asymmetric response of volatility to positive and negative shocks to stock prices; return volatility tends to decrease when stock prices go up, and vice versa. When d=0, the FIEGARCH model reduces to Nelson's EGARCH, and when d=1, it becomes IEGARCH.

Some researchers have examined the properties of a probability model in which both stock price jumps and autoregressive stochastic volatility shocks are combined --the jumps add to kurtosis, but the kurtosis dies out, while the stochastic volatility shocks, *e.g.* in a GARCH-like model, will persist. Das and Sundaram (1997) examine various combinations of jump and stochastic volatility models, where the kurtosis due to jumps dies out in longer-run return distributions, but where the decrease can be more than offset in the medium-term by kurtosis due to persistence in diffusion-volatility shocks. (In the long run, of course, kurtosis due to both jumps and stochastic volatilities becomes small as the law of large numbers starts to bind).

One problem with the FIEGARCH and jump-diffusion models is that although these models can be made to fit the data well, they don't offer much intuition about the process generating shifts in volatility. Ding and Granger (1996) also point out that a number of other processes can have long memory, including generalized fractionally integrated models arising from aggregation, time-varying coefficient models, and nonlinear models. Granger and Ding also propose a multi-components-of-volatility model where it is possible to give some economic interpretation to the components, *e.g.* heterogeneous investors or information arrivals, as in Andersen and Bollerslev (1997).

If the kurtosis and skewness are due to stochastic variation in stock price volatility which cannot be hedged away, then valuation formulas for the options will have to make some strong assumptions, *e.g.* Merton's (1976) formula assumed that jump risk was entirely diversifiable. Moreover, if the option valuation continues to use a certainty-equivalent approach (*i.e.* just insert point estimates of parameters in the valuation formula), then we need to be careful, both because of the "high" standard errors in estimating the higher order moments and the nonlinearity of the option value in those moments (or the parameters of the jump-diffusion-stochastic volatility models that generate the moments).

3.2 Subordinated Stochastic Process Models

Stock returns are measured in calendar time, *e.g.* the return over a day or a week. The idea behind representing these returns as a subordinated stochastic process is that the "effective" or "operational" time scale on which stock prices evolve is not the same as calendar time. For example, one could think of stock prices evolving "more quickly" when a lot of information about the stock arrives[10]. Clark (1973) suggested that trading volume could serve as a proxy for this speed of evolution of stock prices with respect to calendar time; Mandelbrot and Taylor (1967) had suggested "transaction time."

Defining the time scale for evolution of stock prices in operational time units as τ, and calendar time as t, the time change from calendar to operational time can be represented by the function $\tau = \theta(t)\,|_{t\geq 0}$. The stochastic function $\theta(t)\,|_{t\geq 0}$ is known as the subordinating or directing process, and the observed return for a given stock (or exchange rate or whatever) becomes $R(\theta(t))\,|_{t\geq 0}$. If the stochastic time rescaling $\theta(t)\,|_{t\geq 0}$ is modeled judiciously, for example in terms of the number of trades or trading volume [11] for the stock, then the stock's price may be assumed to follow a geometric Brownian motion with respect to trading time but have fat tails in calendar time returns. In this event, the probability distribution function

[10] When we say that someone is "getting much older" in times of hard work or "hard living," we are positing that aging follows a subordinated stochastic process.

[11] Marsh and Rock (1986) analyzed tick-by-tick stock price changes and found that imbalances between the *number of trades* on the ask and bid sides seemed to explain those changes better than the imbalances between their *volumes*.

for the subordinated process $R(\theta(t))\,|_{t\geq 0}$ will be a mixture of normals with unknown parameters for the subordinating process, and with a form that is unknown except in special cases.

The idea of differentiating between operational time and calendar time in economics goes back at least to Burns and Mitchell's famous work in which they developed a National Bureau of Economic Research (NBER) time scale for business cycles. A business cycle was treated as a distinct unit of "economic time" which was broken into four roughly equal lengths in months. The phase averaged data corresponding to their time transformation was calculated as the average of the observations over the months that fall into the relevant stage of the cycle. Stock (1987) tested for the NBER and two other general forms of time deformation in income, money stock, population, the inflation rate, and short-term commercial paper rates. His results were mixed --- they suggested that there wasn't a common time scale transformation for these business cycle variables. The results were more encouraging for stock returns and exchange rates, possibly because time transformation variables like number of trades are in fact the "correct" ones to use if stock price and exchange rate movements are partly endogenous to the process of trading itself.

If we think of $\theta(t)\,|_{t\geq 0}$ as the number of operational time "steps" in period t, then $\mathrm{var}\left[R(\theta(t))\middle|\,\theta = v\right] = v\sigma_R^2$, where σ_R^2 is the variance of return in operational time, and v is the number of operational time "steps" (*e.g.* number of trades) in clock period t. If it is assumed for illustration that the directing process $\theta(t)\,|_{t\geq 0}$ and the return $R(\theta)$ are independent, then the unconditional (subordinated) return variance is $E(\theta)\sigma_R^2 + \mu_R^2\sigma_\theta^2$, where σ_θ^2 is the variance of the directing process $\theta(t)\,|_{t\geq 0}$ (*e.g.* the number of trades per period) and μ_R is the expected return in trade time. If $\mu_R \approx 0$ then the subordinated return volatility depends primarily upon the expected number of operational time steps in each period and not the variance of $\theta(t)\,|_{t\geq 0}$. But the kurtosis of the subordinated distribution does increase with the variance of $\theta(t)\,|_{t\geq 0}$; in fact, Mandelbrot and Taylor (1967) showed that if the stock return process is subordinate to a symmetric stable distribution and the directing process is also generated by a stable process, it will have an infinite variance.

The subordinated model is appealing because it seems economically sensible to associate the directing process with trading intensity. It is tempting to model the distribution of returns in trade time as Gaussian and then specify a model of trading intensity, *i.e.* the directing process, to make the subordinated model fit the data, because the Gaussian process leads to analytically tractable solutions. Yet this seems counter-productive if it is necessary to distort the subordinating process just to accommodate the Gaussian return assumption. Fortunately, it seems that trading time models do seem to fit reasonably well with the Gaussian model to explain observed stock returns; it will be interesting to test whether such subordinated models explain apparent departures of option prices from Black-Scholes.

3.3 Scaling--Fractal Models

Mandelbrot (1997) argues that geometric Brownian motion is a poor model for stock prices and other financial time series and, by extension, for pricing options. He characterizes the Brownian model as a model of "mild randomness" which is not capable of describing the "wild randomness" in actual data. The features of this wild randomness in stock returns include "...the irregular alternation of quiet periods and bursts of volatility"; sharp discontinuities in the time series of returns, which in practice are often referred to as "crashes," "corrections," "prices gaps" etc.; the concentration or bunching of these discontinuities; long-term dependence[12], and the fat tails in observed return distributions. Indeed, Mandelbrot (1997, p. 26) writes that "soon after 1900, Bachelier himself saw that the data are nonGaussian and statistically dependent." Mandelbrot proposes a fractal/multifractal model to account for the "wild random" features of stock returns. The multifractal model is closely related to the subordinated model just discussed in Section 3.2.

To briefly explain the multifractal model, let R_{t+n} be the logarithmic change in stock price over the n periods from time t to time $t+n$, $R_{t+n} = \ln S_{t+n} - \ln S_t$. In continuous time, define this return over an instant as $R(t)$. The stochastic process for the return $R(t)$ is called multifractal if it is stationary and satisfies $E(|R(t)|^q) = c(q)t^{\tau(q)+1}$ for all $t \in T, q \in Q,$ where T and Q are intervals on the real line, and $c(q)$ and $\tau(q)$ are functions with domain Q (Mandelbrot, Fisher, and Calvert (1997a)). The function $\tau(q)$ is the scaling function of the multifractal process. The scaling function in the multifractal process is

[12] Mandelbrot and Wallis (1968) dubbed the long-term dependence "Noah and Joseph Effects," alluding to Biblical references to the bunching of periods of "lean-times" and "good-times" (or cows).

restricted so that returns in the model display statistical self-similarity. To understand self-similarity, imagine that time is re-scaled so that an interval of time Δt becomes $c\Delta t$. Then if the return in the newly scaled time $R(ct)$ is "self-affine," which is roughly the same as self-similar: $R(ct) \stackrel{d}{=} c^{H(c)}R(t)$, where $H(c)$ is a random function of c, and the scaling function can be written as $\tau(q) = Hq - 1$.

The self-similarity of returns over different time scales is in the spirit of fractal geometry. Mandelbrot (1997) proposes that these restrictions are critical in overcoming a deficiency in models like those in Clark (1973) and Blattberg and Gonedes (1974). Both these papers, like the more recent jump-diffusion – stochastic volatility models discussed above, give solutions for the subordinated distribution of stock returns, *e.g.* it is student-*t* in Blattberg and Gonedes. Mandelbrot points out that the student-*t* form of the subordinated distribution does not scale, *i.e.* if the subordinated distribution is student-*t* for daily data, it won't be student-*t* for monthly data. This failure of scaling presumably assumes that it is a prominent feature of the data, *i.e.* that while "mild randomness" is not a good description for daily data, it might be reasonable for monthly data.

The scaling function $\tau(q)$ can be interpreted within the subordinated model where clock time is scaled into trading time: To see this, represent the stock return $R(t)$ as a compound/subordinated stochastic process $R(t) \equiv B_H[\theta(t)]$ where $B_H(t)$ is a fractional Brownian motion and $\theta(t)$ is the stochastic directing process. As in Section 3.2 above, t denotes clock time and $\theta(t)$ is operational or trading time. Assuming that $\theta(t)$ is independent of $B_H(t)$[13], then $R(t)$ can be represented as multifractal with $E(|R(t)|^q) = E[\theta(t)^{Hq}]E[|B_H(1)|^q]$, where $c_\theta(q) \equiv c(Hq)E[|B_H(1)|^q]$ and $\tau(q) \equiv \tau_\theta(Hq) - 1$ (see Mandelbrot, Fisher, and Calvert (1997a)).

The moments of a stock return series that follows a multifractal model can be easily cast in terms of a subordinated model. They depend upon the directing process that transforms clock time to operational or trading time (where the directing process is restricted to satisfy self-similarity in the returns). The q-th moment of returns will exist if and only if the directing process $\theta(t)$ has a moment of order Hq. Notice

[13] A big assumption: it is easy to imagine that when there is more "action" in a stock, the price change process also has different characteristics.

the convenient feature that the multifractal model accounts for long-term memory in the absolute value of stock returns by construction[14]; hence it can satisfy the martingale property for stock returns but still fit long memory and fat tails. The GARCH class of models only produce long-term memory[15] for the fractionally-integrated GARCH (FIGARCH) models unless a lot of GARCH parameters are introduced. Mandelbrot argues for the multifractal model over FIGARCH because the latter doesn't possess scale consistency.

An alternative approach to modeling the stochastic process for stock prices is to view them as being produced by complex systems or chaos, where the complexity is modeled as, say, feedback in stock prices when some investors are using dynamic trading strategies, or other heterogeneity is present. Platen and Schweizer (1994) have already proposed that feedback effects in prices can, when portfolio insurance-type strategies are being used, induce a smile pattern of implied volatilities consistent with that observed in practice. We include in this category the models of the stock market as a cooperative self-organizing system with critical threshold points at which large shifts in stock prices occur. Perhaps the investor biases, informational asymmetry, and the like which cause feedback in these complex systems are most plausible when the market is under stress. Thus, we discuss elements of this approach in Section 5 when we turn to the issue of systemic risk in the use of derivatives.

3.4 Nonparametric Binomial Tree Representations

A fourth approach is to express the conditional distribution for a stock's price directly in terms of the tree diagram that is consistent with the implied probability distributions fitted to the prices of options on the stock. This approach is an extension of Rubinstein's (1994) approach, see Jackwerth and Rubinstein (1996) and Jackwerth (1996) for references. It is useful where the objective is to value one set of options, *e.g.* OTC

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[14] Mandelbrot, Fisher, and Calvet (1997a, fn 8) reference a paper by Taqqu (1975) establishing long memory in the absolute values of the increments of a fractional Brownian motion process.

[15] It was noted earlier that the long-memory property seems to be much weaker in post World War II returns than prior to that.

options, so that they are consistent with the observed pricing of a set of traded options on the same instrument.

In the implied probability/binomial tree approach, the conditional volatility of stock prices is expressed as a function of the stock price at every node on the tree. The advantage or disadvantage, depending on one's view, is that the implied binomial tree for stock prices depends on critical identification assumptions. The implied tree must contain all paths for future possible stock prices, including those reflecting both permanent and transitory movements in volatility, whereas the autoregressive time-series models for volatility, such as GARCH, retain only the forecasted "permanent" component of volatility at option maturity (at least for European options). On the other hand, the shifts in conditional volatility are driven by the stock price and thus can be assumed hedgeable.

4. Generalizing the Option Valuation Formulas for Higher Order Moments

There is still the issue of valuing the options for generalized stock price distributions, though obviously it is straightforward for the implied tree approach. Backus, Foresi, and Wu (1997) propose using a Gram-Charlier expansion as a more parsimonious representation than the mixture of normal densities that can be used to incorporate jumps and stochastic volatilities. The Gram-Charlier expansion generates an approximate density function for stock returns in which terms incorporating kurtosis and skewness are added to the standard normal density.

Let S_t be the price of a stock at time t, and R_{t+n} be the logarithmic change in the stock price over the n periods extending from time t to time $t+n$, $R_{t+n} = \ln S_{t+n} - \ln S_t$. Denote the expected n period return as μ_n, the variance as σ_n, the skewness as γ_{1n}, and the kurtosis as γ_{2n}. Defining the standardized return $w \equiv \dfrac{R_{t+n} - \mu_n}{\sigma_n}$, its approximate density given by the Gram-Charlier expansion is:

$$f(w) = \phi(w) - \gamma_{1n} \frac{1}{3!} D^3 \phi(w) + \gamma_{2n} \frac{1}{4!} D^4 \phi(w), \tag{1}$$

where $\phi(w) = (2\pi)^{-1/2} \exp(-w^2/2)$ is the standard normal density and $D^j\phi$ is the j-th derivative of ϕ. Denote the price of a call option on the stock at time t when the stock price is S_t, expiring n periods later, as C_{nt}. Backus, Foresi, and Wu (1997) show that C_{nt} is approximately:

$$C_{nt} \approx S_t\phi(d) - Ke^{-r_n n}\phi(d-\sigma_n) + S_t\phi(d)\sigma_n\left[\frac{\gamma_{1n}}{3!}(2\sigma_n - d) - \frac{\gamma_{2n}}{4!}(1 - d^2 + 3d\sigma_n - 3\sigma_n^2)\right]. \tag{2}$$

In (2), the call price is expressed as the Black-Scholes value plus terms involving the skewness γ_{1n}, and the kurtosis γ_{2n} of the stock's return. The kurtosis lowers the value of at-the-money options (d=0) relative to Black-Scholes and increases the values of deep in-the-money and deep out-of-the-money options. The intuition is straightforward: if an option is deep out-of-the-money, kurtosis in stock prices means there is a higher chance that a big stock price increase can occur which will put the option in-the-money at maturity. And vice versa for deep-in-the-money options. Backus, Foresi, and Wu show that the approximation (2) seems to be a reasonable approximation to the call option values computed for the mixture-of-normals probability density for the jump-diffusion model,[16] particularly when jumps are frequent and skewness is small.

Note that since the focus is on the third and fourth moments of stock price distributions, a method-of-moments procedure could be useful here --- indeed, it is one instance in which choice of the appropriate moments to use in the method of moments is implicit in the desired analysis itself!

5 Endogenous Stock Price Dynamics and the Possibility of Systemic Risk Posed by Derivatives

Whether the widespread use of derivatives increases systemic risk is controversial. The concerns are usually along the lines expressed in the recent Barron's article "So You're Insuring Against Market Risk With Options," November 24, 1997. The thesis there can be summarized as follows: Dealers who write put options typically delta hedge, but they will be exposed to gamma risk if it is not feasible for them to adjust their hedge positions sufficiently as stock prices move. Particularly if liquidity "dries up," *i.e.* there are discontinuities in prices and/or trading costs become very high if stock prices move sharply, this gamma risk is potentially large relative to dealer capital.

The possibility that dealers will go bankrupt because they improperly manage the gamma risk of their OTC and traded derivatives positions poses a credit risk for their counter-parties[17], but that credit risk does not *per se* cause a systemic risk. In principle it is still not a systemic risk even if the defaults would be correlated across dealers because asset prices are correlated and dealers tend to be short gamma, or the like. The underlying concern rather seems to be that "if the market plunges, dealers will be obliged to rebalance their portfolios by selling stocks, just to reduce their exposure to further declines. This requirement to sell into a decline will tend to increase the volatility of the market and render it more liable to self-reinforcing spirals" (Barron's Online, page 9).[18]

The empirical evidence so far doesn't give much credence to these concerns. In the last decade, as derivatives use has expanded, average stock market and interest rate volatility has arguably been lower than historically, at least in the U.S. since the 1920s[19]. Moreover, even in October and November 1997, as prices moved sharply on many of the world's stock markets, derivatives seem, if anything, to have helped markets function smoothly. The Brady Commission report generally exonerated portfolio insurance as a "cause" if not a contributor to the 1987 stock market crash, and discussed the importance of the Federal Reserve's role in injecting liquidity when settlement mismatches led to bankruptcy rumors. And, of course, derivatives use and gamma risk were unknown when the stock market crashed in 1929-1930.

Still, the "worriers" tend to argue that rare events like market meltdowns won't, by definition, be witnessed very often, and in any case October 1987 came close. In this case, we can only turn to theory.

[16] Note that Merton assumes that jump risk is diversifiable when he derives the call option pricing formula for the jump-diffusion model. This assumption is harder to justify for market index options and currency options.

[17] Just as a disastrous wheat crop might increase the price of wheat, *i.e.* a change in relative prices in the economy, but doesn't per se cause inflation. Moreover, the counter-parties should include any gamma risk in their assessments of default risk.

[18] Of course, rebalancing is also necessary as stock prices go up, and thus could cause something of an upward spiraling in stock prices; but this is usually neglected because present-day authors usually implicitly or explicitly think the market is overvalued --- explicitly, in the case of the Barron's article: "...such a spiral could set off the crash to which the market's extreme overvaluation makes it vulnerable."

Does theory shed any light on potential systemic risk arising from derivatives, then? The focus has been on use of portfolio insurance strategies, either through put options on the market or dynamic trading of market baskets of stocks. Hedgers who use dynamic trading strategies to put a floor under their portfolio returns, or the dealers who are delta hedging the puts sold to the hedgers, will sell stocks as the price goes down, and buy as prices go up. Brennan and Schwartz (1989) examined what will happen when an increasing number of "representative" investors switch to portfolio insurance strategies. They assumed that stock prices exogenously follow a continuous sample path process and both the portfolio insurers and remaining investors who sell them insurance are perfectly informed about the strategies of the other. In their model, investors care only about their consumption at a future point in time. With no confusion engendered by the hedgers' short-term strategies, and no other source of illiquidity, portfolio insurance has very little effect on steady-state stock market volatility. For example, even if 20% of the representative investors are insuring their portfolios and their coefficient of relative risk aversion is 4, market volatility is only 7% higher than it would be if there were no portfolio insurance. Brennan and Schwartz did note, however, that even in the environment they assumed, trading volume could be substantial, *e.g.* with 10% portfolio insurance and a coefficient of relative risk aversion of 2, each 1% drop in stock prices would cause a turnover of approximately 0.2% of market value (this turnover is comparable with a recent average daily turnover of approximately 0.28% on the NYSE and 0.13% on the Tokyo Stock Exchange, 1-st Section).

Grossman (1988) suggested that when investors cannot accurately estimate the extent of portfolio insurance, there is the potential for illiquidity problems. Gennotte and Leland (1990) showed that selling by portfolio insurers as prices go down can be magnified if uninformed investors, who can't observe the extent of portfolio insurance, rationally infer that part of the selling is instead done by informed investors. In both these papers, if the extent of portfolio insurance were made explicit by the use of put options, investors would be able to infer the amount of selling by option writers who were adjusting their delta hedges (and thus reducing gamma risk). In this case, portfolio insurance wouldn't cause stock prices to "gap down." It may be, as Andrew Smithers reports in the Barron's article, that "...the problems involved in obtaining accurate data on the options markets...are staggering" (p. 4).

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[19] It is often observed that the historical volatility of the U.S. market (or the risk premium) is not necessarily a good predictor of the future because there is a survival bias, *i.e.* countries such as Russia and China should be included in the ex ante predictions.

Or it may be that derivatives strategies will eventually be found to cause self-reinforcing spirals in stock price for reasons other than informational incompleteness ---- even if the dog hasn't yet barked, except possibly when it was kept in the dark in October 1987, there will always be reasons why it *might* bark in the future! Frey and Stremme (1997) simply assumed a positive feedback from current prices to agent's expectations about future prices, and in so doing they found that the impact of dynamic hedging strategies was considerably higher than in the Brennan and Schwartz base case (particularly if the hedging were concentrated on a small number of contracts). Basak and Cuoco (1997) show that restricted participation by individuals in the stock market can substantially affect asset pricing, and it would probably substantially affect the impact of dynamic hedging if that were included in their model. Hansen and Stremme (1997) introduce OTC options trading by a large, informed trader, and find that this informed trader's desire to manipulate prices can cause market equilibrium to collapse. A smile that is consistent with those observed in practice also emerges in their analysis. This research endogenizing price behavior is interesting, and it takes our understanding a long way beyond the Black-Scholes model, but at the same time it sometimes underscores the debt owed to the Black, Scholes, and Merton work. For example, in the richer structural environment, Hansen and Stremme still report that "...the informed trader's valuation of the option can be identified as the Black-Scholes price of a different derivative security, written on the (expected) true value of the underlying asset!" (1997, p. 23).

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