CIRJE-F-49

Check Your Partners' Behavior by Randomization:
New Efficiency Results on Repeated Games with Imperfect Monitoring

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May 1999

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October 7, 1998

This version: May 27, 1999

Abstract

Randomization is an effective way of extracting information from a limited number of observations, as random auditing shows. We employ this idea to support efficient outcomes in repeated games with imperfect monitoring, when information is severely limited. In particular, we show that efficiency can be improved when the players randomize and condition their future actions *both* on the signal *and* their actions. Firstly, we show that in a version of Radner, Myerson and Maskin's example of inefficient partnership, efficiency can be achieved by such (i.e., *privately* mixed strategy) equilibria. Secondly, we show that the folk theorem under imperfect public monitoring can be extended to the case with a small signal space by means of mixed strategy equilibria with communication. In particular, we show that *for generic symmetric games with at least four players, we can drop the Fudenberg-Levine-Maskin condition on the number of actions and signals altogether and prove the folk theorem under the same condition as in the perfect monitoring case.*

* This research was originated in a stimulus conversation with Tadashi Sekiguchi. I would like to thank the saminar participants at Osaka University, Tokyo University and LSE, especially A. Matsui, V. Bhaskar, and J. Moore for helpful comments and discussion. Financial support from CIRJE at University of Tokyo is also gratefully acknowledged.

1. Introduction

Even when a limited number of cases can be investigated, IRS is able to discipline a large number of taxpayers by randomizing whom to audit. Likewise, factory managers employ random sampling for quality control, when only a small number of their products are to be examined. Randomization is an effective way of extracting information from limited observations. We will show that this idea can be fruitfully employed to support efficient outcomes in a long-term relationship.

More specifically, we will show that efficiency can be achieved in repeated games with imperfect public monitoring, even when the amount of available information is severely limited. This is done by introducing a new class of mixed strategies that have not been employed in the previous literature on repeated games with imperfect public monitoring. This class of games has attracted much attention since Green and Porter's seminal work on cartel enforcement under demand uncertainty (1984). A general technique to analyze a certain class of equilibria (see below) was introduced by Abreu, Pearce and Stacchetti (1990) and Fudenberg, Levine and Maskin (1994) (FLM hereafter) proved the Folk Theorem.

All those works restrict attention to a special class of equilibria (*perfect public equilibria*) where *the players ignore what they have done* and condition their actions only on the history of the publicly observed signals. FLM showed that this class of equilibria is rich enough to support any mutually beneficial outcomes (i.e., to prove the Folk Theorem) *when the signal takes on sufficiently many values relative to the number of available actions.* In particular, in generic symmetric games with K actions for each player and L different values for the public signal, the FLM Folk Theorem obtains when $2K-1 \leq L$. While one can reasonably argue that there are many variables which can potentially serve as the signals (so that L is large), one can also tell equally convincing stories that the players can potentially take a variety of actions (K is also large). In short, it would be fair to say that the FLM Folk Theorem is applicable to a wide range of situations, but an important class of games, where information is rather limited, is not covered by their results. Examples of such games include repeated partnership where the project in each period either succeeds or fails (L=2). The present paper provides the Folk Theorem for such a class of situations.

The main idea can roughly be explained by the analogy to the random sampling technique for quality control. The manager randomly picks up one of the products assembled by the workers and examines its quality. To provide incentives for the workers, their compensation should depend not only on the quality of the sampled product (the value of the signal) but also on whose product was examined (the action of the manager). Similarly in repeated games, if the players randomize over different actions and the future payoffs can depend *both* on the realized signal *and* actions, rich information can be generated to discipline the players.

This idea is embodied in two sets of results. Firstly, we show that efficiency can sometimes be improved when the players employ *(mixed) private strategies.* As we have seen, nearly all existing works, including FLM, restrict attention to *public strategies,* where the playrs' actions depend only on the history of the publicly observable signal. In contrast, in mixed private strategies each player's action also depends on the history of her own action. We re-examine Radner, Myerson and Maskin's example (1986), which shows that all equilibria in *public* strategies are bounded away from the Pareto frontier, when the signal space is small relative to the action space. We show that in a version of their model efficiency can be (approximately) achieved when the players utilize private strategies.

Secondly, we introduce communication to the game. At the end of each period, each player announces which action she has just taken. The players can tell a lie but we will construct equilibria where the players voluntarily reveal the true actions. We look at the public equilibria in such a game, where the players condition their actions both on the history of public signal and communication[1]. In such an equilibrium, the players' future payoffs can depend both on the signal and actions, and a rich class of information can be generated by randomization (as in the random sampling technique for quality control discussed above). In particular, we show that *for generic symmetric games with at least four players, we can drop the FLM condition on the number of actions and signals altogether and prove the folk theorem under the same condition as in the perfect monitoring case.* This result is particularly useful in various applications of the folk theorem, because quite often we do not have a clear idea about the number of actions and signals when formalizing a given economic problem as a mathematical model.

The paper is organized as follows. Section 2 introduces the basic model of repeated games with imperfect public monitoring. Section 3 shows that efficiency can

[1] Communication has been introduced in repeated games with imperfect *private* monitoring, where each player privately receive a different signal about the past actions (Ben-Porath and Kahneman (1996), Compte (1998) and Kandori and Matsushima (1998)). The main role of communication in those works, which mainly look at pure strategies, is to coordinate the players' actions, rather than to generate more information.

be achieved in a version of Radner, Myerson and Maskin's example, when the players employ private strategies. Section 4 introduces communication and proves the Folk Theorem in the situation where the signal space is relatively small compared to the action space. In section 5 we examine the way to provide strict incentives for truth-telling.

2. The Basic Model

A group of players i=1,2, ..., N repeatedly play a stage game over time, t=0,1,.... The stage game payoff function for player i is given by $u_i(a_i, \omega)$, where $a_i \in A_i$ is player i's action and $\omega \in \Omega$ is publicly observable signal. We assume that each player's action is not observable to other players. Given an action profile $a \in A = A_1 \times \cdots \times A_N$, the probability of ω is denoted by $p(\omega \mid a)$ (We assume that Ω and A are finite sets). Player i's *expected* stage game payoff is defined by

$$g_i(a) = \sum_{\omega \in \Omega} u_i(a_i, \omega) p(\omega | a).$$

Note that this formulation, which is standard in the literature, makes sure that the realized payoff of player i (that is, u_i) conveys no more information than his own action a_i and the public signal ω do. This stage game is repeatedly played, and the *average payoff* of player i is given by

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t g_i(a(t)),$$

where a(t) refers to the action profile in period t and $\delta \in (0, 1)$ is the discount factor. We denote a mixed strategy profile by $\alpha \in \Delta = \Delta_1 \times \cdots \times \Delta_N$, and abuse the notation to represent the corresponding expected payoff and signal distribution by $g_i(\alpha)$ and $p(\omega \mid \alpha)$. Finally, we define the stage game payoff profile by $g=(g_1,\ldots,g_N)$.

Next we will review the Fudernberg-Levine algorithm (1994) to compute the perfect public equilibrium payoffs in the limit (as the discount factor tends to unity). This method allows us to determine the limit value set in the repeated game by means of a set of associated static contract problems. For a given welfare weight $\lambda \in \Re^N$, define $k(\lambda)$ by

$$\sup_{\alpha\in\Delta,\, x:\Omega\to\Re^N} \lambda(g(\alpha)+E[x(\omega)|\alpha])$$
$$s.t.\quad (IC)\quad \forall i \forall a'_i\ g_i(\alpha)+E[x_i(\omega)|\alpha] \geq g_i(a'_i,\alpha_{-i})+E[x_i(\omega)|a'_i,\alpha_{-i}]$$
$$(B)\quad \forall\omega\ \lambda x(\omega)\leq 0.$$

This could be interpreted as an optimal contract design problem for the stage game, where the social planner tries to maximize the social welfare by means of a side payment scheme x, whose value depends on the publicly observable signal ω. Condition (B) can be regarded as a budget balancing requirement under the given welfare weight, and (IC) shows that each player is optimizing under the given side payment scheme x. Let $D(\lambda) = \{v\in\Re^N \mid \lambda v \leq k(\lambda)\}$ and denote the set of perfect public equilibrium payoffs under discount factor δ by $V(\delta)$. A perfect public equilibrium is a sequential equilibrium where each player's action at time t depends only on the realization of past signals $\omega(s)$, $s<t$. Then the following is true.

Lemma 1 (Fudenberg and Levine (1994)).

$$\lim_{\delta\to 1} V(\delta) = \bigcap_{\lambda\neq 0} D(\lambda),$$

if the right hand side has non-empty interior.

We will employ an extension of this result to the case where communication is allowed. In each period, each player i takes an action, observes ω, and then[2] (simultaneously with other players) announces a message m_i. As we examine the case where the players reveal their action, we assume $m_i\in A_i$. The players can tell a lie but we will construct equilibria where they voluntarily tell the truth. Let $m_i=c_i(a_i,\omega)$ be player i's communication strategy in the stage game. The set of player i's communication strategies is denoted by C_i and we let $C=C_1\times\cdots\times C_N$. For a given welfare weight $\lambda\in\Re^N$, we define $k^c(\lambda)$ by

[2] Alternatively, we can assume that the players communicate *before* the realization of ω. Our results in Section 4 and 5 show that the players truthfully reveal their actions after each realization of ω. Then, under the same arrangements the players have incentives for truth-telling before observing ω. Hence our results hold for either specification.

$$\sup_{\alpha\in\Delta,\, c\in C,\, x:\Omega\times A\to\Re^N} \lambda(g(\alpha)+E[x|\alpha,c])$$

$$\text{s.t.}\quad (IC^c)\quad \forall i \forall a'_i \forall c'_i\ g_i(\alpha)+E[x_i|\alpha,c]\geq g_i(a'_i,\alpha_{-i})+E[x_i|a'_i,\alpha_{-i},c'_i,c_{-i}]$$

$$(B^c)\quad \forall\omega\forall m\ \lambda x(\omega,m)\leq 0,$$

where $E[x|\alpha,c]$ is the expected value of $x(\omega,m)$ under (α,c). Define $D^c(\lambda) = \{v\in\Re^N \mid \lambda v\leq k^c(\lambda)\}$ and let $V^c(\delta)$ be the set of perfect semipublic equilibrium payoffs in the game with communication under discount factor δ. A perfect semipublic equilibrium is a sequential equilibrium where each player's action at time t depends only on the realization of past signals $\omega(s)$ and messages $m(s)$, $s<t$. Then we have the following.

Lemma 2.

$$\lim_{\delta\to 1} V^c(\delta)\supset\bigcap_{\lambda\neq 0} D^c(\lambda),$$

if the right hand side has non-empty interior.

Proof: If we redefine the stage game payoff and the signal by $g'(a,c)=g(a)$ and $\omega'=(\omega,m)$, the stage game has a finite action space $A\times C$ and signal space $\Omega\times A$, where the signal distribution is given by

$$p'(\omega,m|a,c)=\begin{cases} p(\omega|a) & \text{if } \forall i\, m_i=c_i(a_i,\omega)\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2 is derived by applying Lemma 1 to this stage game [3].

3. Private Strategies and Efficiency

In this section we look at repeated prisoners' dilemma game with imperfect monitoring. There are two players, and two actions, C and D, are available for each palyer. We also assume that the publicly observable signal ω takes on two values, X or Y. The expected stage game payoff profiles are summarized by the following table.

[3] If we used mixed communication strategies in the definition of $k^c(\lambda)$, we would have equality rather than set inclusion $\supset$. For our purpose this weaker version suffices.

	C	D
C	1, 1	-h, 1+d
D	1+d, -h	0, 0

Each entry of the table denotes the column player's payoff followed by the row player's payoff. We assume that this is a *prisoners' dilemma* game; d, h > 0 (D is dominant) and d-h < 1 ((C,C) is efficient, that is, it is not Pareto-dominated by the equal (public) randomization between (C,D) and (D,C)). This is a simplified version of the model examined by Radner, Myerson and Maskin (1986).

Consider the following information structure; $0 < p(X \mid C,C) < 1$, $0 < p(X \mid D,D) < 1$, and $p(X \mid C,D)=p(X \mid D,C)=0$. The last equalities represent a "moving support" assumption, but note that this does not help to support the efficient payoff profile (1,1) by public strategies, as we will show later. Also note that the prisoners' dilemma payoffs in the above table can be generated by suitable choices of realized payoffs $u_i(a_i, \omega)$ so as to satisfy

$$1 = u_i(C,X)\ p(X \mid C,C) + u_i(C,Y)\ p(Y \mid C,C),$$
$$-h = u_i(C,Y),$$
$$1+d = u_i(D,Y), \text{ and}$$
$$0 = u_i(D,X)\ p(X \mid D,D) + u_i(D,Y)\ p(Y \mid D,D).$$

Consider the following *private* strategy. In the initial period t=0, each player mixes between C and D. Action C is chosen with a (large) probability $r \in (0, 1)$. If the realization of the signal at the end of the current period is X *and* she played D, then she will play D forever. Otherwise, the player repeats the same action plan as in the initial period. Note well that (i) the players are using their private information (their actions) and (ii) thanks to the assumption $p(X \mid C,D)=p(X \mid D,C)=0$, when a player chose D and observes X, it is common knowledge that the other player also chose D (and of course is observing X). Point (ii) thus ensures that *the players can fully coordinate to trigger punishment (defecting forever)*, as in the usual trigger strategy equilibrium. The equilibrium conditions are

$$v = (1-\delta)(r - (1-r)h) + \delta v \tag{3.1}$$

$$v = (1-\delta)r(1+d) + \delta[1 - (1-r)p(X \mid D,D)]v \tag{3.2}$$

Equation (3.1) represents the average payoff at the beginning of the initial period when the player under consideration plays C (while the opponent is employing the above strategy). In this case, the current payoff is either 1 or $-h$ depending on the opponent's action, and punishment is surely avoided (as defection is triggered if and only if *both* payers play D and the signal is X). On the other hand, equation (3.2) shows the payoff when the player chooses D. The current payoff is either 1+d or 0, and the punishment will be triggered when the opponent also chooses D and the signal is X. This happens with probability $(1-r)p(X \mid D,D)$, so with the complementary probability, the player will enjoy the original average payoff v at the beginning of the following period. Equation (3.1) and (3.2), taken together, imply that the player is just indifferent between choosing C and D (the condition for a mixed strategy equilibrium).

From (3.1), we have

$$v = r - (1-r)h. \tag{3.3}$$

Also, by (3.1) and (3.2) we get

$$(1-\delta)(rd + (1-r)h) = \delta(1-r)\, p(X \mid D,D)v.$$

This and (3.3) result in a quadratic equation in r;

$$(1-\delta)((d-h)r + h) = \delta(1-r)\, p(X \mid D,D)\, ((1+h)r - h). \tag{3.4}$$

We will show that there is a root of this equation in (0, 1) which tends to 1 as $\delta \to 1$. Equation (3.3) then shows that, as r tends to 1, the average payoff tends to 1, the payoff from full cooperation. This leads us to the following result.

Proposition 1. *In the repeated prisoners' dilemma game defined above, there is a private equilibrium that approximately attains fully efficient average payoff (=1) as the discount factor tends to unity, while any perfect public equilibrium average payoff is bounded away from 1.*

Proof. To show the efficiency of the private equilibrium given above, we need to prove that a root of equation (3.4) lies in (0, 1) and tends to unity as δ tends to 1. At r=1, the left hand side of (3.4) is strictly positive ($(1-\delta)d > 0$) but the right hand side is equal to zero. Now let r be any number $r' \in (h/(1+h), 1)$ and let δ tends to 1. The left hand side of (3.4) tends to zero, while the right hand side tends to

$$(1\text{-}r')\ p(X \mid D,D)\ ((1+h)r' - 1) > 0.$$

Thus equation (3.4) has a solution in (r', 1) as δ tends to 1, where r' is any number close to 1. Lemma 1 can be used to show that any perfect public equilibrium payoff is bounded away from 1. The details are similar to Radner, Myerson and Maskin (1986) and therefore omitted.

4. Randomization, Communication and Efficiency

In the last section, we showed that efficiency could be improved if the players randomize and condition their future action plans *both* on the signal *and* their actions. However, as their actions are private information, it is in general difficult to coordinate their future action plans properly when they employ such (i.e., *private*) strategies[4]. This coordination problem was resolved by the special information structure in the previous example, but in general it is hard to cope with. To overcome such difficulties, we introduce communication in this section. In each period, each player takes an action, observes ω, and then[5] (simultaneously with other players) announces which action she has just chosen. The players can tell a lie but we will construct equilibria where they voluntarily tell the truth. To implement this idea we need at least three players, which will be assumed throughout this section.

First let us introduce some notation. For a given pair of players i and j, we denote their opponents' action profile by $a_{-ij} \in A_{-ij} \equiv \prod_{k \neq i,j} A_k$, and their mixed action profile $\alpha_{-ij} \in \Delta_{-ij}$ is defined similarly. Let $R_i(j,\alpha_{-i})$ be the matrix whose rows and columns are indexed by a_i and (ω,a_{-ij}) respectively with its $(a_i,(\omega,a_{-ij}))$-element given by

$$\sum_{a_j} p(\omega|a) \prod_{k \neq i} \alpha_k(a_k), \qquad (4.1)$$

[4] The same difficulty arises when there is no publicly observable information and each player privately observes a distinct signal. The study of such a class, i.e., repeated games with imperfect private monitoring, is in its rudimentary stage. See, for example, Bhaskar and van Damme (1999), Mailath and Morris (1998) and Sekiguchi (1997).
[5] For an alternative specification of timing, see footnote 2.

and define $Q_{ij}(\alpha)$ by

$$Q_{ij}(\alpha)=\begin{pmatrix} R_i(j,\alpha_{-i}) \\ R_j(i,\alpha_{-j}) \end{pmatrix}.$$

To avoid possible welfare loss, we will provide incentives for any given pair of players i and j by efficiently allocating their future payoffs. That is, when we punish one player, we reward the other so as to avoid punishing them simultaneously, which entails welfare loss. As we also need to induce truth-telling, we let their future payoffs depend on the signal and what *others* announce, (ω,a_{-ij}). The a_i-th row of matrix $Q_{ij}(\alpha)$ represents the distribution of this information; when player i chooses action a_i and other players employ mixed actions α_{-i}, then (ω,a_{-ij}) realizes with probability given by (4.1) (provided that other players tell the truth). The a_j-th row can be interpreted similarly. If $Q_{ij}(\alpha)$ has the maximum row rank, we can statistically detect which of the two players deviated, and thus the efficient punishment discussed above is feasible.

Let $q_{ij}(a_k)$ (k=i,j) and $q_{ij}(\omega,a_{-ij})$ be the a_k-th row and the (ω,a_{-ij})-th column of matrix $Q_{ij}(\alpha)$. The number of the columns is $|\Omega|\times\prod_{k\neq i,j}|A_k|$, but as we have

$$\sum_{\omega} q_{ij}(\omega,a_{-ij})=\prod_{k\neq i,j}\alpha_k(a_k)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$$

for each a_{-ij}, at most $(|\Omega|-1)\prod_{k\neq i,j}|A_k|+1$ columns can be independent. The number of rows are $|A_i|+|A_j|$, but as

$$\sum_{a_i} q_{ij}(a_i)\alpha_i(a_i)=\sum_{a_j} q_{ij}(a_j)\alpha_j(a_j)=(\cdots,\sum_{a_i,a_j} p(\omega|a)\prod_{k=1}^{N}\alpha_k(a_k),\cdots),$$

at most $|A_i|+|A_j|-1$ rows can be independent. Therefore, a necessary condition for $Q_{ij}(\alpha)$ to have the maximum row rank is

$$(|\Omega|-1)\prod_{k\neq i,j}|A_k|\geq|A_i|+|A_j|-2. \tag{4.2}$$

The next lemma shows that this is almost sufficient.

Lemma 3. *Under (4.2), the following are true for a generic choice of signal distribution $p(\omega | a)$.*

(i) For any $\varepsilon>0$ and any pure strategy profile a, there is a mixed strategy profile α such that $||g(a)-g(\alpha)||<\varepsilon$ and $Q_{ij}(\alpha)$ has the maximum row rank for any pair i, j.

(ii) Take any $\varepsilon>0$ and any profile (a_i, α_{-i}) where a_i is a best response against α_{-i}. Then there is a profile (a'_i, α'_{-i}) such that $||g_i(a_i, \alpha_{-i})- g_i(a'_i, \alpha'_{-i})||<\varepsilon$, a'_i is a best response against α'_{-i}, and $R_j(i, (a'_i, \alpha'_{-ij}))$ has full row rank for each $j\neq i$.

Proof. (i) For each pair i, j, fix a profile $\alpha(i,j)=(a^0_i,a^0_j,\alpha^0_{-ij})$, where α^0_{-ij} is completely mixed. Then, each element of matrix $Q_{ij}(\alpha(i,j))$ is $p(\omega | a')$ times a positive constant, with a distinct (ω,a') for each element (except that the a^0_i-th and a^0_j-th rows are identical). Therefore, $Q_{ij}(\alpha(i,j))$ has the maximum row rank under (4.2) in an open and dense set, say P(i,j), in the space of all possible signal distributions $p(\cdot | \cdot)$. Thus in an open and dense set $\bigcap_{i\neq j} P(i,j)$, for each pair i,j, $Q_{ij}(\alpha(i,j))$ has the maximum row rank.

Take any $(|A_i|+|A_j|-1)\times(|A_i|+|A_j|-1)$ submatrix of $Q_{ij}(\alpha)$ that is nonsingular when $\alpha=\alpha(i,j)$, and denote its determinant by $D_{ij}(\alpha)$. Since $D_{ij}(\alpha)$ is a polynomial in α and nonzero for $\alpha=\alpha(i,j)$, we conclude that it is nonzero on an open and dense set, say $\Delta(i,j)$, in the space of mixed strategy profiles. Thus, for a generic choice of the signal distribution, $Q_{ij}(\alpha)$ has the maximum row rank for each i,j on an open and dense set $\bigcap_{i\neq j} \Delta(i,j)$. Claim (i) then follows from the continuity of the stage payoff function g.

(ii) By a similar argument as above, we can show that for a generic choice of the signal distribution, there is an open and dense set Δ' such that for any pair i,j and any a''_i, matrix $R_j(i, (a''_i, \alpha''_{-ij}))$ has full row rank if $\alpha''\in\Delta'$. Choose $\alpha'\in\Delta'$ so that α'_{-i} is sufficiently close to the given profile α_{-i}. Then, we have $||g_i(a_i, \alpha_{-i})- g_i(a'_i, \alpha'_{-i})||<\varepsilon$ by the Berge's theorem, and $R_j(i, (a'_i, \alpha'_{-ij}))$ has full row rank for each $j\neq i$.

Lemma 3 assures that we can generate sufficiently rich information by means of randomization and communication as long as condition (4.2) is satisfied. Combining this and Lemma 2 leads us to the following folk theorem. Let $\underline{v}_i$ be player i's minimax value $\underline{v}_i = \max_{a_i} \min_{\alpha_{-ij}} g_i(a_i, \alpha_{-i})$ and let $(\underline{a}_i, \underline{\alpha}^i_{-i})$ be the profile that achieves this value.

Define the set of feasible and individually rational payoff set by $V^*=\{v \in cog(A) \mid \forall i\ v_i \geq \underline{v}_i\}$, where cog(A) is the convex hull of pure strategy stage payoffs.

Proposition 2 (Folk Theorem). *Suppose $N \geq 3$ and V^* has non empty interior. Under (4.2), for a generic choice of the signal distribution, any feasible and individually rational payoff profile can be approximately achieved by a sequential equilibrium with communication, if the discount factor is sufficiently close to unity:*

$$\lim_{\delta \to 1} V^c(\delta) \supset V^*.$$

Remark: Condition (4.2), $(|\Omega|-1)\prod_{k \neq i,j} |A_k| \geq |A_i|+|A_j|-2$, is much weaker than the FLM condition for their folk theorem, $|\Omega| \geq |A_i|+|A_j|-1$, especially when there are many players. We will show (Corollaries 1 and 2) that our condition is automatically satisfied for a wide class of games with at least four players.

Proof. In the light of Lemma 2, it is sufficient to show $\bigcap_{\lambda \neq 0} D^c(\lambda) \supset V^*$. Fix $\lambda \neq 0$ and define $I(\lambda)=\{i \mid \lambda_i \neq 0\}$. We will determine $D^c(\lambda)$ in the following three possible cases:

(a) $I(\lambda)=\{i\}$ for some i and $\lambda_i>0$.

(b) $I(\lambda)=\{i\}$ for some i and $\lambda_i<0$.

(c) $|I(\lambda)| \geq 2$.

In case (a), let a* be the profile that maximizes player i's stage payoff. By Lemma 3 (ii), for any given $\varepsilon>0$, we can always find a profile (a'_i, α'_{-i}) in such a way that $||g_i(a^*)-g_i(a'_i,\alpha'_{-i})||<\varepsilon$, a'_i is a best response against α'_{-i}, and $R_j(i, (a'_i, \alpha'_{-ij}))$ has full row rank for each $j \neq i$. Let us define the side payment scheme in the definition of $D^c(\lambda)$ by $x_i \equiv 0$ and $x_j(\omega,a)=y_j(\omega,a_{-ij})$ for $j \neq i$. Note that each player's side payment is independent of her announcement, so that telling truth is an optimal strategy. Then, condition (IC^c) is satisfied at (a'_i, α'_{-i}) for player i, because a'_i is a best response against α'_{-i} and $x_i \equiv 0$. To satisfy condition (IC^c) for player $j \neq i$, we will show that we can choose x_j so that

$$\forall a_j \quad v_j = g_j(a_j, a'_i, \alpha'_{-ij}) + E[x_j \mid a_j, a'_i, \alpha'_{-ij}]$$

for some v_j. This can be rewritten as

$$\begin{pmatrix} \vdots \\ v_j - g_j(a_j, a'_i, \alpha'_{-ij}) \\ \vdots \end{pmatrix} = R_j(i,(a'_i, \alpha'_{-ij})) \begin{pmatrix} \vdots \\ y_j(\omega, a_{-ij}) \\ \vdots \end{pmatrix},$$

and it has a solution in y_j because $R_j(i, (a'_i, \alpha'_{-ij}))$ has full row rank. Hence we have x which satisfies (IC^c) and (B^c) with equality to support profile (a'_i, α'_{-ij}). Since we have $\| g_i(a^*)-g_i(a'_i,\alpha'_{-i}) \| < \varepsilon$ and ε can be arbitrarily small, we have $k^c(\lambda)=\lambda g(a^*)=\max_{a\in A}\lambda g(a)$.

Case (b) is similar and we can show that minimax point for player i, $g(\underline{a}_i,\underline{\alpha}^i_{-i})$, is approximately supported by a side payment scheme with $\lambda x\equiv 0$, so that we have $k^c(\lambda)\geq\lambda g(\underline{a}_i,\underline{\alpha}^i_{-i})=\lambda_i\underline{v}_i$.

In case (c), let a* be a maximizer of $\lambda g(a)$. Then, by Lemma 3 (i), for any $\varepsilon>0$, we can find a mixed strategy profile α such that $\| g(a^*)-g(\alpha) \| < \varepsilon$ and $Q_{ij}(\alpha)$ has the maximum row rank for all pair i, j. We will show that α can be supported by a side payment scheme satisfying $\lambda x\equiv 0$. When $j\notin I(\lambda)$, choose any $i\neq j$ and find $x_j(\omega,a)=y_j(\omega,a_{-ij})$ so that j has an incentive to follow α_j. This can be done exactly in the same way as in case (a). For any pair $i,j \in I(\lambda)$, fix any (v_i,v_j) and find $y^{ij}_k(\omega,a_{-ij})$ (k=i,j) such that

$$\forall a_i \quad v_i = g_i(a_i,\alpha_{-i}) + E[y^{ij}_i \mid a_i,\alpha_{-i}], \tag{4.3}$$

$$\forall a_j \quad v_j = g_j(a_j,\alpha_{-j}) + E[y^{ij}_j \mid a_j,\alpha_{-j}] \text{ and}$$

$$\lambda_i y^{ij}_i + \lambda_j y^{ij}_j \equiv 0. \tag{4.4}$$

This set of conditions is equivalent to

$$\forall a_i \quad v_i - g_i(a_i,\alpha_{-i}) = E[y^{ij}_i \mid a_i,\alpha_{-i}] \text{ and}$$

$$\forall a_j \quad -(\lambda_i/\lambda_j)[v_j - g_j(a_j,\alpha_{-j})] = E[y^{ij}_i \mid a_i,\alpha_{-i}],$$

and by a similar argument as in case (a), this system is shown to have a solution in y^{ij}_i, as $Q_{ij}(\alpha)$ has the maximum row rank. Now for $i \in I(\lambda)$, define x_i to be the average of y^{ij}_i: $x_i = \frac{1}{\# I(\lambda)-1}\sum_{j\neq i, j\in I(\lambda)} y^{ij}_i$. Then, by (4.3) and (4.4), we have x which satisfies (IC^c) and (B^c) with equality to support α. As we have $\| g(a^*)-g(\alpha) \| < \varepsilon$ and ε can be arbitrarily small, we have $k^c(\lambda)=\lambda g(a^*)=\max_{a\in A}\lambda g(a)$.

From the conclusions of the three cases, $D^c(\lambda)$ is $\{v \mid \lambda v \le \max_{a\in A}\lambda g(a)\}$ in cases (a) and (c) and it contains $\{v \mid v_i \ge \underline{v}_i\}$ in case (b). Hence we have $\bigcap_{\lambda\neq 0} D^c(\lambda) \supset V^*$.

As an application of Proposition 2, consider a game where the players have an equal number of actions. Then we can show that the folk theorem generically holds for such a game *no matter how small the signal space may be*, as long as there are at least four players.

Corollary 1. *Consider a game where the players have an equal number of actions, and suppose $N\ge 4$ and V^* has non-empty interior. Then, for a generic choice of the signal distribution, any feasible and individually rational payoff profile can be approximately achieved by a sequential equilibrium with communication, if the discount factor is sufficiently close to unity.*

Proof. As the realized payoff for player i depends only on his action and the signal, there is no strategic interaction when $|\Omega|=1$. Then V^* is a singleton set with the Pareto efficient Nash equilibrium payoff of the stage game if $|\Omega|=1$ so that the statement is trivially true. When $|\Omega|\ge 2$ we invoke Proposition 1, and condition (4.2) reduces to $(|\Omega|-1)K^{N-2} \ge 2K-2$, where K is the number of available actions for each player. When $|\Omega|\ge 2$ and $N\ge 4$, we have $(|\Omega|-1)K^{N-2} - (2K-2) \ge K^2 - 2K + 2 \ge (K-1)^2+1 \ge 0$.

We will now prove a similar result for symmetric games. As some restrictions are imposed on the information structure (i.e., on matrices Q_{ij}) by the symmetry of $p(\omega \mid a)$ (it should be invariant with respect to any permutation of the elements of a), we have to check if the generic full rank properties (Lemma 3) hold for symmetric games. The answer turns out to be positive, and we obtain the following result.

Corollary 2. *Consider symmetric games, and suppose $N\ge 4$ and V^* has non-empty interior. Then, for a generic choice of the (symmetric) signal distribution, any feasible and individually rational payoff profile can be approximately achieved in a sequential equilibrium with communication, if the discount factor is sufficiently close to unity.*

Proof. See appendix.

5. Strict Incentives for Truth-Telling

So far we constructed equilibria where each player's expected future payoff is independent of what she announces, thereby providing her a *weak* incentive to tell the truth. In this section we modify the equilibria to provide *strict* incentive for truth-telling. Our construction is based on the following idea. Suppose the realization of some random variable y is commonly observable to i and j, but its probability distribution p(y) is only known to i. Is it possible for j to induce i to reveal the true distribution p? The answer is yes, and j can let i announce a distribution, say $r(\cdot)$, and when y is realized provide i with payoff $2r(y)-\Sigma_{y'}r(y')^2$. Then, i's expected payoff is

$$\sum_{y}[2r(y)-\sum_{y'}r(y')^2]p(y),$$

and this concave quadratic form in r is uniquely maximized at the point where the first order conditions $\forall y\ 2p(y)-2r(y)=0$ are satisfied. Hence announcing the truth $r = p$ is the unique optimal for i.

In the context of repeated games, y corresponds to a_{-i}, and under our assumption that each player announce her action after observing ω, player i knows that the probability distribution of a_{-i} is given by $\Pr(a_{-i} \mid a_i,\omega)$, which corresponds to p in the above construction. Suppose i receives payoff

$$f_i(\omega,m) = 2\Pr(m_{-i} \mid m_i,\omega) - \sum_{a'_{-i}\in A_{-i}}\Pr(a'_{-i} \mid m_i,\omega)^2\,, \tag{5.1}$$

which depends on the announcements $m\in A$. When other players tell the truth ($m_{-i}=a_{-i}$), the expected payoff $E[f_i \mid a_i,\omega]$ is maximized when $\Pr(\cdot \mid m_i, \omega)$ is the true distribution of a_{-i}, as announcing m_i amounts to be choosing a particular distribution $r(\cdot)= \Pr(\cdot \mid m_i, \omega)$ in the above construction. Hence truth-telling $m_i=a_i$ is an optimal announcement. Furthermore, if there is no other $a'_i\neq a_i$ which induces the same distribution, $\Pr(a_{-i} \mid a_i, \omega) = \Pr(a_{-i} \mid a'_i, \omega)$ for all a_{-i}, then announcing the truth is the *unique* optimal.

Hence strict incentives for truth-telling can be provided if we modify the variation of future payoff, which is represented by x_i in Lemma 2, to incorporate a small term proportional to f_i, provided that we have

$$\forall\omega\forall i\forall(a'_i, a''_i) \text{ such that } a'_i\neq a''_i \quad \exists a_{-i}\ \Pr(a_{-i} \mid a'_i, \omega) \neq \Pr(a_{-i} \mid a''_i, \omega). \tag{5.1}$$

Condition (5.1) requires that the players' actions are not independent *conditional on* ω. Note that even though each player independently mixes so that a_{-i} and a_i are *ex ante* independent, the posterior distribution of a_{-i} usually depends on a_i once player i observes the signal ω. The main idea here is to exploit this statistical dependence of private actions to provide strict incentives for truth-telling. As we will show below, it is easy to see that the reqired condition (5.1) is generically satisfied.

Proposition 3. *Suppose that $N\geq 3$, V^* has non-empty interior, and communication is allowed. Under (4.2), for a generic choice of the signal distribution, any feasible and individually rational payoff profile can be approximately achieved by a sequential equilibrium where all players have strict incentives to announce their true actions, if the discount factor is sufficiently close to unity.*

Proof. Fix any $\lambda\neq 0$ and let (α^*,x) be the pair identified in the proof of Proposition 2 that approximately achieves $k^c(\lambda)$. Instead of x_i, we will introduce a new side payment scheme x'_i, which is represented as

$$x'_i(\omega,m) = y_i(\omega,m_{-ij}) + \rho(f_i(\omega,m) - B_i),$$

where ρ is a small positive number and B_i is a real number to satisfy $\lambda_i(f_i(\omega,m) - B_i)\leq 0$ for all (ω,m). Note that y_i is independent of what i and another (suitably chosen) player j announce, as is the case with x_i in the proof of Proposition 2. If condition (5.1) is satisfied at α^*, each player i has a strict incentive for truth telling under x'_i, as we have seen above. By similar arguments as in the proof of Proposition 2, we can show the existence of y_i such that (i) x'_i satisfies (IC^c) in Section 2 and (ii) $\lambda y(\omega,a)=0$ for all (ω,a). (The details can be found in Appendix.) Then, by choosing ρ sufficiently small we can make $\lambda x'(\omega,a)$ an arbitrarily small negative number for all (ω,a). Hence we conclude that (α^*,x') also approximates $k^c(\lambda)$.

Condition (5.1) is satisfied if for all ω, i, and (a'_i, a''_i) such that $a'_i\neq a''_i$, we can find (a^0_{-i}, a^1_{-i}) such that

$$\frac{\Pr(a^0_{-i}|a'_i,\omega)}{\Pr(a^1_{-i}|a'_i,\omega)} \neq \frac{\Pr(a^0_{-i}|a''_i,\omega)}{\Pr(a^1_{-i}|a''_i,\omega)}.$$

As the conditional probability is given by

$$\Pr(a_{-i}|a_i,\omega)=\frac{P(\omega|a)\alpha^*_{-j}(a_{-i})}{\sum_{a'_{-i}}\Pr(\omega|a_i,a'_{-i})\alpha^*_{-j}(a'_{-i})},$$

where $\alpha^*_{-i}(a_{-i})$ is the probability of a_{-i} under α^*, condition (5.1) is satisfied if for all ω, i, and (a'_i, a''_i) such that $a'_i \neq a''_i$, we can find (a^0_{-i}, a^1_{-i}) such that

$$\alpha^*_{-i}(a^k_{-i}) > 0,\ k=0,1 \text{ and} \tag{5.2}$$

$$\frac{p(\omega|a'_i,a^0_{-i})}{p(\omega|a'_i,a^1_{-i})}\neq\frac{p(\omega|a''_i,a^0_{-i})}{p(\omega|a''_i,a^1_{-i})}. \tag{5.3}$$

Clearly, condition (5.3) is satisfied generically as $a'_i \neq a''_i$ and $a^0_{-i} \neq a^1_{-i}$.

We can also show that condition (5.1) is also satisfied generically in the space of symmetric signal distributions, so that providing strict incentives for truth-telling for symmetric games are also possible:

Corollary 3. *Consider symmetric games, and suppose that $N \geq 4$, V^* has non-empty interior, and communication is allowed. Then for a generic choice of the (symmetric) signal distribution any feasible and individually rational payoff profile can be approximately achieved by a sequential equilibrium where all players have strict incentives to announce their true actions, if the discount factor is sufficiently close to unity.*

Proof. We can choose a^0_{-i} and a^1_{-i} satisfying (5.2) in such a way that they consists of different combinations of actions. Then, the symmetry does not require $p(\omega | a'_i, a^0_{-i})$ to be identical to neither $p(\omega | a'_i, a^1_{-i})$ or $p(\omega | a''_i, a^0_{-i})$, because each of (a'_i, a^0_{-i}), (a'_i, a^1_{-i}) and (a''_i, a^0_{-i}) consists of a distinct combination of actions (Recall $a'_i \neq a''_i$). Hence (5.3) is satisfied generically in the space of symmetric signal distributions.

Appendix

A. Proof of Corollary 2.

We only need to check if the statements in Lemma 3 hold under symmetric signal distributions. Hence Corollary 2 is shown by the following lemma.

Lemma 4. *When $N \geq 4$, the following are true for a generic choice of symmetric signal distribution $p(\omega \mid a)$.*
(i) For any $\varepsilon > 0$ and any pure strategy profile a, there is a mixed strategy profile α such that $\| g(a) - g(\alpha) \| < \varepsilon$ and $Q_{ij}(\alpha)$ has the maximum row rank for any pair i, j.
(ii) Take any $\varepsilon > 0$ and any profile (a_i, α_{-i}) where a_i is a best response against α_{-i}. Then there is a profile (a'_i, α'_{-i}) such that $\| g_i(a_i, \alpha_{-i}) - g_i(a'_i, \alpha'_{-i}) \| < \varepsilon$, a'_i is a best response against α'_{-i}, and $R_j(i, (a'_i, \alpha'_{-ij}))$ has full row rank for each $j \neq i$.

Proof. Let $A_i=\{1,2,\ldots,K\}$ for each i and $\Omega=\{\omega^1,\ldots,\omega^L\}$. We assume $K,L \geq 2$, because Corollary 2 trivially holds when K=1 or L=1 (for the latter case see the proof of Corollary 1).

Part (i): For any pair of players i and j, define $\alpha'(i,j)$ to be the profile where (i) i and j are taking pure action 1 and K respectively and (ii) other players mix all actions with an equal probability. Fix any symmetric distribution where

$$p(\omega^1 | a) = \begin{cases} 1 & \text{if } \#\{h | a_h = k\} \text{ is even for all } k = 2,\ldots,K \\ 0 & \text{otherwise,} \end{cases}$$

and consider the (partial) table of $p(\omega \mid a)$'s in Figure A.1, where n and m are any distinct players other than i and j. The rows and columns of the table are indexed by (a_i,a_j) and (ω,a_{-ij}) respectively. This forms a lower-triangular matrix (call it B) with 1's on the diagonal, because we have the following:

The elements in area (A) are equal to 0 because $\#\{h \mid a_h=a_m\}=1$ and $a_m \geq 2$.
The elements in area (B) are equal to 0 because $\#\{h \mid a_h=K\}=3$.
The elements in area (C) are equal to 0 because $\#\{h \mid a_h=K\}=1$.
The elements in area (D) are equal to 0 because $\#\{h \mid a_h=a_n\}=1$ and $a_n \geq 2$.

*** Figure A.1 about here ***

Hence matrix B is non-singular because its determinant is equal to 1 (the product of its diagonal elements). Note that $Q_{ij}(\alpha'(i,j))$ has a sumbatrix equal to $K^{-(N-2)}B$, because at $\alpha'(i,j)$ each a_{-ij} is taken with an equal probability $K^{-(N-2)}$. Therefore, $Q_{ij}(\alpha'(i,j))$ has the maximum row rank for the particular distribution given above. As the determinant of B is polynomial in $p(\omega \mid a)$, we conclude that $Q_{ij}(\alpha'(i,j))$ has the maximum row rank in the open and dense set P'(i,j) in the space of symmetric signal distributions. The rest of the proof is identical to Lemma 3 (i).

Part (ii): For any pair of players i and j, conider (a''_i, α^*_{-ij}) where a''_i is any fixed pure strategy and each player $n \neq i,j$ mixes all actions with an equal probability. Choose any symmetric distribution with

$$p(\omega^1 | a) = \begin{cases} 0 & \text{if } \exists k \; \#\{h | a_h = k\} = 1 \\ 1 & \text{otherwise,} \end{cases}$$

*** Figure A.2 about here ***

and then we obtain a nonsingular matrix B' of $p(\omega \mid a)$'s in Figure A.2, where m is any player other than i and j. By a similar argument as in part (i), $R_j(i, (a''_i, \alpha^*_{-ij}))$ is shown to have a submatrix equal to $K^{-(N-2)}B'$, and therefore it has full row rank in an open and dense set $P(i,j, a''_i)$ in the space of symmetric signal distributions. Then, in open and dense set $\bigcap_{i,j,a''_i} P(i,j,a''_i)$, we can follow the same argument as in the proof of Lemma 3 (ii).

B. Construction of the side payment schemes in the proof of Proposition 3.

We use Lemma 2 to show the folk theorem, by demonstrating $\bigcap_{\lambda \neq 0} D^c(\lambda) \supset V^*$.

Fix $\lambda \neq 0$ and define $I(\lambda)=\{i \mid \lambda_i \neq 0\}$. As in the Proof of Proposition 2, we need to determine $D^c(\lambda)$ in the three possible cases: (a) $I(\lambda)=\{i\}$ for some i and $\lambda_i>0$, (b) $I(\lambda)=\{i\}$ for some i and $\lambda_i<0$, and (c) $|I(\lambda)| \geq 2$.

Here, we consider only case (c), as other cases are similar. Let a* be a maximizer of $\lambda g(a)$. Then, by Lemma 3 (i), for any $\varepsilon>0$, we can find a mixed strategy profile α such that $||g(a^*)-g(\alpha)||<\varepsilon$ and $Q_{ij}(\alpha)$ has the maximum row rank for all pair i, j. We will show that α can be supported by a side payment scheme that satisfies $-\varepsilon<\lambda x'(\omega,a)<0$ for all (ω,a). When $i\notin I(\lambda)$, let f_i be given by (5.1). Choose any other player $j\neq i$ and consider a function $y_i(\omega,m_{-ij})$. We now define the side payment scheme for i by $x'_i(\omega,m_{-i})=y_i(\omega,m_{-ij})+\rho(f_i(\omega,m_{-i})-B_i)$ for some $\rho>0$ and $B_i\in\Re$. Function y_i is to be determined to satisfy

$$\forall a_i \quad v_i = g_i(a_i, \alpha_{-i}) + E[x_i(\omega,a_{-i}) \mid a_i, \alpha_{-i}] \tag{B.1}$$

for some v_i. This can be rewritten as

$$\begin{pmatrix} \vdots \\ v_i - g_i(a_i,\alpha_{-i}) - E[\rho(f_i - B_i)|a_i,\alpha_{-i}] \\ \vdots \end{pmatrix} = R_i(j,\alpha_{-i}) \begin{pmatrix} \vdots \\ y_i(\omega,a_{-ij}) \\ \vdots \end{pmatrix},$$

and it has a solution in y_i because $R_i(j, \alpha_{-i})$ has full row rank (as $Q_{ij}(\alpha)$ has the maximum row rank). Under x'_i, player i has an incentive to tell the truth after any action a_i and signal ω (because term y_i is independent of what she says and term f_i rewards truth-telling). Given truth telling, (B.1) ensures that player i is indifferent to take any action so that she has an incentive to follow the designated mixed strategy α_i.

For any pair $i,j \in I(\lambda)$, fix any (v_i,v_j) and find $y^{ij}_k(\omega,a_{-ij})$ $(k=i,j)$ such that

$$\forall a_i \quad v_i = g_i(a_i,\alpha_{-i}) + E[y^{ij}_i+\rho(f_i-B_i) \mid a_i,\alpha_{-i}], \tag{B.2}$$

$$\forall a_j \quad v_j = g_j(a_j,\alpha_{-j}) + E[y^{ij}_j+\rho(f_j-B_j) \mid a_j,\alpha_{-j}] \text{ and}$$

$$\lambda_i y^{ij}_i + \lambda_j y^{ij}_j \equiv 0. \tag{B.3}$$

This set of conditions is equivalent to

$$\forall a_i \quad v_i-g_i(a_i,\alpha_{-i}) - E[\rho(f_i-B_i) \mid a_i,\alpha_{-i}] = E[y^{ij}_i \mid a_i,\alpha_{-i}] \text{ and}$$

$$\forall a_j \quad -(\lambda_i/\lambda_j)\{v_j-g_j(a_j,\alpha_{-j}) - E[\rho(f_j-B_j) \mid a_j,\alpha_{-j}]\} = E[y^{ij}_i \mid a_i,\alpha_{-i}],$$

and by a similar argument as above, this system is shown to have a solution in y^{ij}_i, as $Q_{ij}(\alpha)$ has the maximum row rank. Now for $i \in I(\lambda)$, define x'_i by $x'_i = \frac{1}{\#I(\lambda)-1} \sum_{j \neq i, j \in I(\lambda)} y_i^{ij} + \rho(f_i - B_i)$. Then, by the same argument as above, we can deduce from (B.2) that x' satisfies (IC^c). From (B.3), we have $\lambda x' = \Sigma_i \lambda_i \rho(f_i - B_i)$. By choosing B_i suitably and making ρ a small number, we can ensure $-\varepsilon < \lambda x'(\omega,a) < 0$ for all (ω,a). Hence we have found (α,x') which satisfies (IC^c) and (B^c) with the property $\|g(a^*)-(g(\alpha)+\lambda E[x'|\alpha])\| < 2\varepsilon$. As ε can be arbitrarily small, we have $k^c(\lambda)=\lambda g(a^*)=\max_{a \in A}\lambda g(a)$, and therefore $D^c(\lambda)=\{v | \lambda v \leq \max_{a \in A}\lambda g(a)\}$ in case (c). By similar arguments, the same is true for case (a) and $D^c(\lambda)=\{v | v_i \geq \underline{v}_i\}$ in case (b). Hence we conclude $\bigcap_{\lambda \neq 0} D^c(\lambda) \supset V^*$.

References

Abreu, D., D. Pearce and E. Stacchetti (1990) "Towards a Theory of Discounted Repeated Games with Imperfect Monitoring", *Econometrica*, 58, 1041-1064.

Ben-Porath, E. and M. Kahneman (1996) "Communication in Repeated Games with Private Monitoring", *Journal of Economic Theory*, 70, 281-297.

Bhaskar, V. and E. van Damme (1999) "Moral Hazard and Private Monitoring", mimeo., University of Essex.

Compte, O. (1998) "Communication in Repeated Games with Imperfect Private Monitoring", *Econometrica*, 66, 597-626.

Fudenberg, D. and D. Levine (1994) "Efficiency and Observability with Long-Run and Short-Run Players", *Journal of Economic Theory*, 62, 103-135.

Fudenberg, D., D. Levine and E. Maskin (1994) "The Folk Theorem with Imperfect Public Information", *Econometrica*, 62, 997-1040.

Green, E. and R. Porter (1984) "Noncooperative Collusion under Imperfect Price Information", *Econometrica*, 52, 87-100.

Kandori, M. and H. Matsushima (1998) "Private Observation, Communication and Collusion", *Econometrica*, 66, 627-652.

Mailath, G. and S. Morris (1998) "Repeated Games with Imperfect Private Monitoring: Notes on a Coordination Perspective", mimeo., University of Pennsylvania.

Radner, R., R. Myerson and E. Maskin (1986) "An Example of a Repeated Partnership Game with Discounting and with Uniformly Inefficient Equilibria", *Review of Economic Studies*, 53, 59-70.

Sekiguchi, T (1997) "Efficiency in Repeated Prisoner's Dilemma with Private Monitoring", *Journal of Economic Theory*, 76, 345-361.

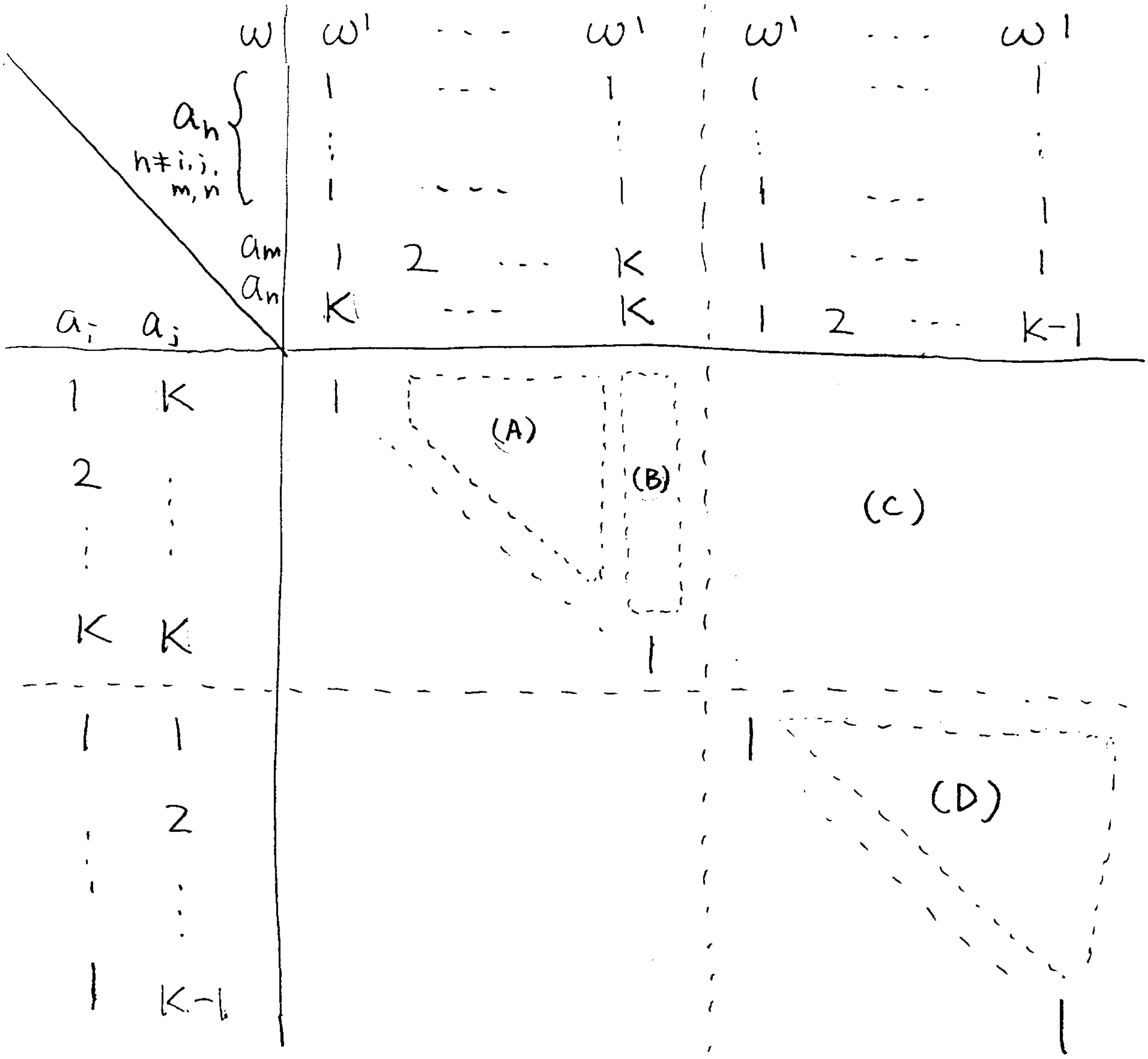

Figure A. 1

(A partial table of p(ω|a).)

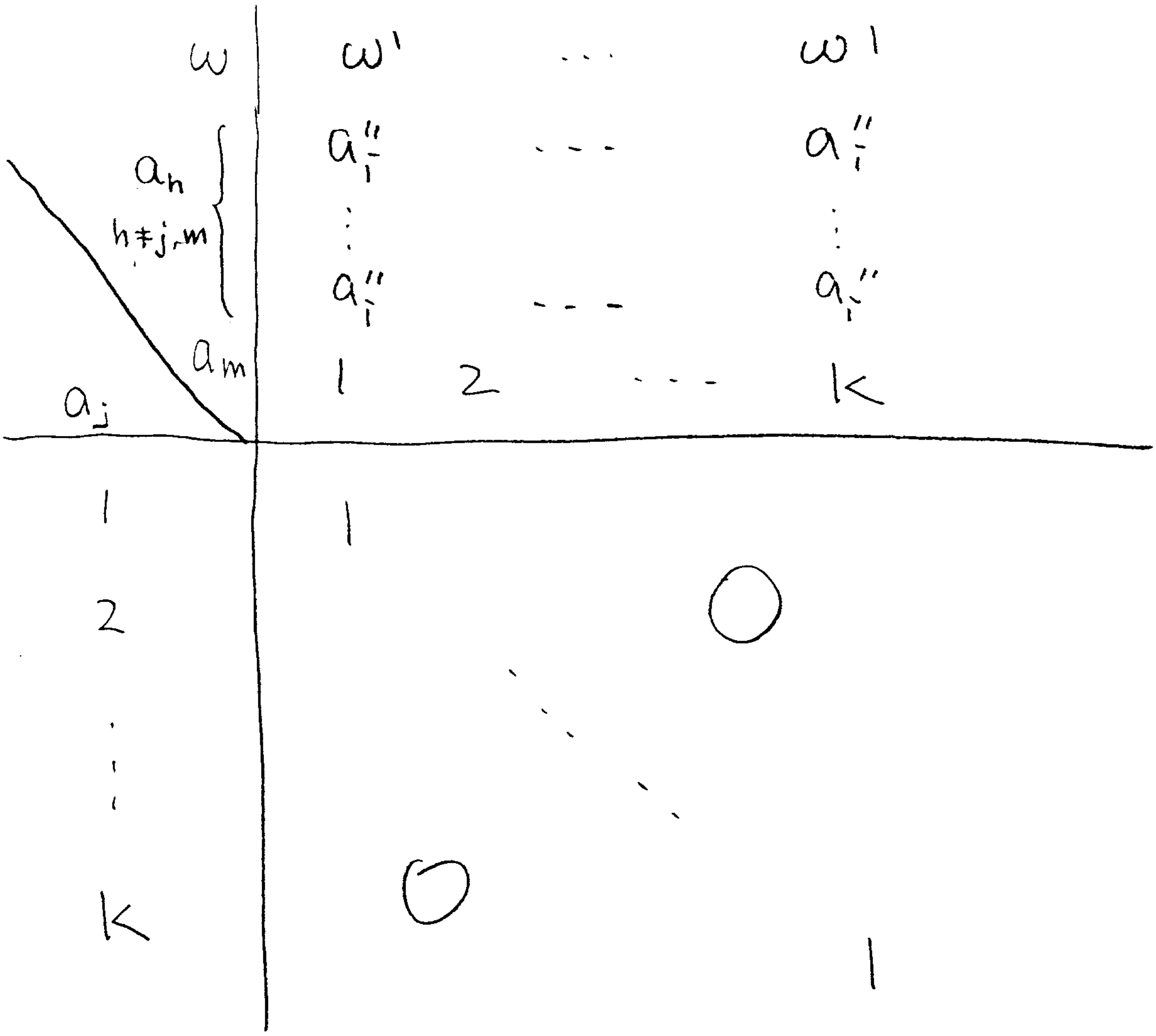

Figure A.2