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On Ergodicity of Some TAR(2) Processes

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Abstract

We give a set of conditions for the geometrical ergodicity and the non-explosiveness of the solutions in the second-order threshold autoregressive (TAR) processes. We also discuss some conditions for the geometrical ergodicity of the second-order simultaneous switching autoregressive (SSAR) processes. Unlike the linear autoregressive processes and the first-order TAR processes, the ergodic and non-explosive regions become quite complicated even in some special TAR processes.

Key Words

Threshold Autoregressive (TAR) Models, Second-Order Processes, Geometrical Ergodicity, Complex Non-explosive Regions, Simultaneous Switching Autoregressive (SSAR) Models.

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1. Introduction

In the statistical time series analysis several nonlinear time series models have been proposed in the past decade. In particular, considerable attention has been paid to the class of threshold autoregressive (TAR) processes, which was systematically investigated by Tong (1990) and some applications have been reported. The TAR processes have been also investigated by Brockwell and Hyndman (1992) in the continuous time framework. Unlike the linear autoregressive models, however, the statistical properties of the second order TAR processes have not been fully investigated mainly due to technical problems involved. The statistical properties of the first order TAR processes have been investigated first by Petruccelli and Woolford (1984), and later by Chen and Tsay (1991) in more details. However, it seems that the necessary and sufficient conditions for the ergodicity have been known only for the first order TAR processes. The main purpose of this paper is to investigate the basic properties of the second-order TAR processes and the second-order SSAR processes.

Let $\{y_t\}$ be a sequence of scalar time series satisfying

$$(1.1) \qquad y_t = \begin{cases} a_1 y_{t-1} + a_2 y_{t-2} + \sigma_1 v_t & \text{if} \quad y_{t-d} \geq 0 \\ b_1 y_{t-1} + b_2 y_{t-2} + \sigma_2 v_t & \text{if} \quad y_{t-d} < 0 \end{cases},$$

where d is a positive integer parameter, and a_i, b_i, σ_i (> 0) $(i = 1, 2)$ are unknown parameters, and $\{v_t\}$ are a sequence of independently and identically distributed (i.i.d.) random variables having an absolutely continuous density with respect to the Lebesgue measure and $E[v_t] = 0$. The second-order threshold autoregressive model given by (1.1) will be denoted as TAR(2:d) in this paper and d is the interger-valued parameter, which is often called the delayed parameter. We also use the notation as TAR(2) for the second order TAR processes. Petrucelli and Woolford (1984) have considered the first-order TAR process when $a_2 = b_2 = 0$ and $d = 1$, which is denoted as TAR(1:1). They have shown that the necessary and sufficient conditions for the geometrical ergodicity are

$$(1.2) \qquad a_1 < 1, b_1 < 1, a_1 b_1 < 1 \ .$$

Chen and Tsay (1991) have extended their results to the TAR(1:d) processes when d is an arbitrary positive integer. The conditions they have obtained for the geometrical ergodicity are more general than (1.2) because they have investigated the first order TAR processes with the positive integer-valued parameter d.

On the other hand, Kunitomo and Sato (1996), and Sato and Kunitomo (1996) have proposed the class of simultaneous switching autoregressive (SSAR) processes, which can be regarded as a natural extension of the TAR processes for some econometric applications. The class of simultaneous switching autoregressive (SSAR) process introduced by Kunitomo and Sato (1996) is slightly different from the threshold autoregressive models. The second-order SSAR model can be written as

$$(1.3) \qquad y_t = \begin{cases} a_1 y_{t-1} + a_2 y_{t-2} + \sigma_1 v_t & \text{if} \quad y_t \geq y_{t-1} \\ b_1 y_{t-1} + b_2 y_{t-2} + \sigma_2 v_t & \text{if} \quad y_t < y_{t-1} \end{cases},$$

where $a_i, b_i, \sigma_i (> 0)$ $(i = 1, 2)$ are unknown parameters, and $\{v_t\}$ are a sequence of independently and identically distributed (i.i.d.) random variables having an absolutely continuous density with respect to the Lebesgue measure and $E[v_t] = 0$. The discrete time series model given by (1.3) will be denoted by SSAR(2) in this paper. By imposing the restrictions on parameters given by

$$\frac{1-a_1}{\sigma_1} = \frac{1-b_1}{\sigma_2} = r_1 \, , \frac{a_2}{\sigma_1} = \frac{b_2}{\sigma_2} = r_2 \, , \tag{1.4}$$

this time series model can be written in a more meaningful way as

$$y_t = \begin{cases} a_1 y_{t-1} + a_2 y_{t-2} + \sigma_1 v_t & \text{if} \quad v_t \geq r_1 y_{t-1} - r_2 y_{t-2} \\ b_1 y_{t-1} + b_2 y_{t-2} + \sigma_2 v_t & \text{if} \quad v_t < r_1 y_{t-1} - r_2 y_{t-2} \end{cases} , \tag{1.5}$$

where r_i $(i = 1, 2)$ are unknown parameters.
Then it has the Markovian representation as

$$\begin{aligned} y_t &= y_{t-1} + [\sigma_1 I(v_t \geq r_1 y_{t-1} - r_2 y_{t-2}) \\ &\quad + \sigma_2 I(v_t < r_1 y_{t-1} - r_2 y_{t-2})][v_t - (r_1 y_{t-1} - r_2 y_{t-2})] \, , \end{aligned} \tag{1.6}$$

where $I(\cdot)$ is the indicator function. When $\sigma_1 = \sigma_2 = \sigma$, then this SSAR process becomes the standard $AR(2)$ model by a re-parametrization. Kunitomo and Sato (1996) have shown that even the simplest SSAR(1) process gives us some explanation and description on an important aspect of the asymmetrical movement of time series in two different (up-ward and down-ward) phases. The ergodicity condition for the SSAR(1) process is the same as (1.2) except the coherency conditions of (1.4).

In Section 2, we give some definitions in the nonlinear time series analysis and a useful lemma for our investigation. In Section 3, we shall give some conditions for the geometrical ergodicity of the TAR(2:d) processes in the leading case. Then we shall discuss some conditions for the geometrical ergodicity of the SSAR(2) and TAR(2:d) processes in the general case in Section 4. Finally in Section 5, we shall give the derivations and the proofs.

2. Some Preliminaries

The first important property of a statistical time series model is whether it is ergodic or not. For the Markovian time series models, the geometrical ergodicity and the related concepts have been developed in the nonlinear time series analysis. For the sake of completeness, we mention to its definition and the drift criterion. For the more precise definitions of related concepts including irreducibility, aperiodicity, and ergodicity of the Markov chains with the general state space, see the Appendix of Tong (1990) or Nummelin (1984).

Definition 1 (Geometrical Ergodicity) :
Let $\{\mathbf{y}_t\}$ be an $m \times 1$ Markovian process with the state space of $\boldsymbol{R}^m$.
(i) $\{\mathbf{y}_t\}$ is geometrically ergodic if there exists a probability measure π on $(\boldsymbol{R}^m, \mathcal{B}(\boldsymbol{R}^m))$,

a positive constant $\rho < 1$, and π-integrable non-negative measurable function $h(\cdot)$ such that

(2.1) $$\|P^n(\boldsymbol{x}, \cdot) - \pi(\cdot)\|_\tau \leq \rho^n h(\boldsymbol{x}) ,$$

where $\| \cdot \|_\tau$ denotes the total variation norm, $\boldsymbol{x} = (x_i)$ is an $m \times 1$ vector, and $P(\boldsymbol{x}, \cdot)$ is the transition probability.
(ii) $\{\mathbf{y_t}\}$ is ϕ-irreducible if for any $\boldsymbol{x} \in \boldsymbol{R}^m$, $A \in \mathcal{B}(\boldsymbol{R}^m)$ with $\phi(A) > 0$ ($\phi(\cdot)$ is a σ-finite measure), and

(2.2) $$\sum_{n=1}^{\infty} P^n(\boldsymbol{x}, A) > 0 .$$

For the geometrical ergodicity of the Markov Chains with the general state, Tjostheim (1990) has given a drift criterion, which will be useful for our purpose and thus reported as Lemma 1. It is an extension of the well-known drift criterion on the Markov chain with the general states. See the Appendix of Tong (1990), for instance.

Lemma 1 *(Tjostheim (1990)) :*
Assume that $\{\boldsymbol{y}_t\}$ is an aperiodic $\phi-$irreducible Markov Chain with the state space of $\boldsymbol{R}^m$ and g is a non-negative measurable function. Then $\{\boldsymbol{y}_t\}$ is geometrically ergodic if there exists a positive integer h, a compact set $\boldsymbol{C}$ satisfying (2.2) with $\phi(C) > 0$, positive constants $\epsilon > 0, M < +\infty$, and $r > 1$ such that

(2.3) $$rE[g(\boldsymbol{y}_{t+h})|\boldsymbol{y}_t = \boldsymbol{y}] \leq g(\boldsymbol{y}) - \epsilon \;\; if \;\; \boldsymbol{y} \in \boldsymbol{C}^c ,$$

and

(2.4) $$E[g(\boldsymbol{y}_{t+h})|\boldsymbol{y}_t = \boldsymbol{y}] \leq M \;\; if \;\; \boldsymbol{y} \in \boldsymbol{C} ,$$

where $\boldsymbol{C}^c$ is the complement of $\boldsymbol{C}$.

Now we introduce another concept on the stability of the solution, which is slightly different from the geometrical ergodicity. Because its conditions are slightly weaker than those in Lemma 1, we use the terminology of *Near Geometrical Ergodicity.* We shall use this concept in this paper due to the technical reason, which will be explained in Section 5. However, the detail of its properties has not been fully exlored. Let $\boldsymbol{Q}$ be the state space of the stochastic process $\{\boldsymbol{y}_t\}$. Then we partition the state space $\boldsymbol{Q}$ into a finite number of subspaces $\boldsymbol{Q}^i$ $(i = 1, \cdots, k)$ such that $\boldsymbol{Q} = \bigcup_{i=1}^{k} \boldsymbol{Q}^i$ and for any t

(2.5) $$1 = \sum_{i=1}^{k} I(\boldsymbol{y}_t \in \boldsymbol{Q}^i) ,$$

where $I(\cdot)$ is the indicator function and k is a positive integer.

Definition 2 (Near Geometrical Ergodicity) :
Let $\{\mathbf{y_t}\}$ be an $m \times 1$ Markovian process with the state space of $\boldsymbol{R}^m$. Then $\{\boldsymbol{y}_t\}$ is near geometrically ergodic if

(i) for any i there exist positive constants r_i $(r_i > 1)$ and positive integers h_i $(i = 1, \cdots, k)$ such that

(2.6) $$r_i E[g(\boldsymbol{y}_{t+h_i}) I(Q^i_{t+h_i}) | \boldsymbol{y}_t = \boldsymbol{y}] \leq g(\boldsymbol{y}) - \epsilon \; if \; \boldsymbol{y} \in Q^i_t \cap \boldsymbol{C}^c \, ,$$

where $I(\boldsymbol{Q}^i_t) = I(\boldsymbol{y}_t \in \boldsymbol{Q}^i)$, and $\boldsymbol{C}$ is a compact set satisfying $\phi(\boldsymbol{C}) > 0$ in (2.2), and (ii) there exist a positive integer h and positive constant M such that

(2.7) $$E[g(\boldsymbol{y}_{t+h}) | \boldsymbol{y}_t = \boldsymbol{y}] \leq M \; if \; \boldsymbol{y} \in \boldsymbol{C} \, .$$

From this definition it is apparent that the near geometrical ergodicity we introduce in this section coincides with the geometrical ergodicity in the nonlinear time series analysis if we can take the common positive integer $h = h_i$ $(1, \cdots, k)$.

3. Conditions for TAR(2) in the leading case

In this section we consider the conditions for the ergodicity of the TAR(2) processes in a special case. For this purpose we shall utilize the Markovian representation of the TAR(2) processes. Let $\boldsymbol{y}_t = (y_t, y_{t-1})'$ be a 2×1 vector of the time series generated by $\{y_t\}$. The TAR(2) process we consider is represented by

(3.1) $$\boldsymbol{y}_t = \begin{cases} \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{D}\sigma_1 v_t & \text{if} \quad \boldsymbol{e}_k' \boldsymbol{y}_{t-1} \geq 0 \\ \boldsymbol{B}\boldsymbol{y}_{t-1} + \boldsymbol{D}\sigma_2 v_t & \text{if} \quad \boldsymbol{e}_k' \boldsymbol{y}_{t-1} < 0 \end{cases} ,$$

where $\boldsymbol{e}_1 = (1, 0)', \boldsymbol{e}_2 = (0, 1)'$ (for $\boldsymbol{e}_k$ with $k = 1, 2$) and $\boldsymbol{D} = (1, 0)'$ are 2×1 constant vectors, and $\boldsymbol{A}$, $\boldsymbol{B}$ are 2×2 coefficient matrices in the Markovian representation given by

(3.2) $$\boldsymbol{A} = \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix} \, , \; \boldsymbol{B} = \begin{pmatrix} b_1 & b_2 \\ 1 & 0 \end{pmatrix} \, .$$

In the rest of this section we consider the leading case for the TAR(2) processes and assume $b_2 = 0$. This is simply because we can obtain general characterizations on the geometrical ergodic regions in this case and the necessary and sufficient conditions for the geometrical ergodicity can be obtainable in most cases. We have to stress that even in this leading case our conditions sometimes become quite complicated and non-standard in comparison with the results known for the TAR(1) processes.

In order to obtain the conditions for the geometrical ergodicity and state our results, we partition the parameter space of (a_1, a_2) into four different regions given by $C_1 = \{a_1 \geq 0, a_2 \geq 0\}$, $C_2 = \{a_1 < 0, a_2 \geq 0\}$, $C_3 = \{a_1 < 0, a_2 < 0\}$, and $C_4 = \{a_1 \geq 0, a_2 < 0\}$, respectively.
Because we set $b_2 = 0$, we need to consider two cases when $b_1 \leq 0$ and $b_1 > 0$. Then it is intuitively obvious that in the latter case we have to restrict the conditions for the geometrical ergodicity when $0 < b_1 < 1$ in order to avoid the possible explosion of the solutions.

3.1 TAR(2:1)

First, we consider the TAR(2:1) process when $b_1 > 0$ and $b_2 = 0$. This is the simplest case in the TAR(2) process in terms of our conditions on the coefficients.
Let

$$D(\boldsymbol{A}) = a_1^2 + 4a_2 , \tag{3.3}$$

which is the discriminant of the characteristic equation for the first phase in (3.1)

$$g_A(\lambda) = \lambda^2 - a_1\lambda - a_2 = 0 . \tag{3.4}$$

Then we present the necessary and sufficient conditions for the geometrical ergodicity. All proofs and derivations of the propositions in this section are given in Section 5.

Condition I :
C_1 : $a_1 + a_2 < 1 , 0 < b_1 < 1$,
C_2 : $0 < b_1 < 1$,
C_3 : $0 < b_1 < 1$,
C_4 : [*either* $a_1 + a_2 < 1$ $(0 \leq a_1 < 2, D(\boldsymbol{A}) \geq 0)$ *or* $D(\boldsymbol{A}) < 0$], $0 < b_1 < 1$.

Proposition 1 : *For the TAR(2:1) process with* $b_1 > 0$ *and* $b_2 = 0$, *the necessary and sufficient conditions for the geometrical ergodicity of* $\{y_t\}$ *are given by* ***Condition I***.

Second, we consider the TAR(2:1) process when $b_1 \leq 0$ and $b_2 = 0$. The conditions in this case become far more complicated than in the first case. In order to deal with some complications involved, we define $\rho(\boldsymbol{A}^k\boldsymbol{B})$ be the non-zero characteristic root of the matrix $\boldsymbol{A}^k\boldsymbol{B}$ for any positive integer k. In the present case

$$\rho(\boldsymbol{A}^k\boldsymbol{B}) = \boldsymbol{e}_1'\boldsymbol{A}^k\boldsymbol{b} ,$$

where $\boldsymbol{b}' = (b_1, 1)$.

Then we give a set of conditions for the geometrical ergodicity and the near geometrical ergodicity of the solution in the present case. The concept of the near geometrical ergodicity of the solution has been given in Definition 2 in Section 2. We have introduced it by a technical reason to which we shall mention in Section 5. We note that it is not needed for the linear time series processes.

Condition II :
C_1 : $a_1 + a_2 < 1$,
C_2 : $\rho(\boldsymbol{AB}) < 1$,
C_3 : $\rho(\boldsymbol{AB}) < 1, \rho(\boldsymbol{A}^2\boldsymbol{B}) < 1$,
C_4 : *either* $[a_1 + a_2 < 1$ $(0 \leq a_1 < 2, D(\boldsymbol{A}) \geq 0)]$ *or* [*there exists a non − negative* k (≥ 3) *such that* $\rho(\boldsymbol{A}^{k-1}\boldsymbol{B}) < 0$ *and* $0 \leq \rho(\boldsymbol{A}^k\boldsymbol{B}) < 1$].

Proposition 2 : *For the TAR(2:1) process with* $b_1 \leq 0$ *and* $b_2 = 0$, *(i) the necessary and sufficient conditions for the geometrical ergodicity of* $\{y_t\}$ *are given by* ***Condition II*** *with* C_1, C_2, *and* C_3, *and (ii) the sufficient conditions for the near geomatrical ergodicity of the solution* $\{y_t\}$ *are given by* ***Condition II*** *with* C_4.

For an illustration, we present two figures of the ergodic regions for the TAR(2:1) processes in Figure 1 and Figure 2, which are based on the simulations of the stochastic processes $\{y_t\}$. Contrary to the ergodic regions for the linear AR(2) models which have been known in the statistical time series analysis (see Brockwell and Davis (1991) for instance), they are often unbounded as we see in these figures. Some of the conditions above can be written more explicitly by using

$$\rho(\boldsymbol{AB}) = a_1b_1 + a_2 \ , \rho(\boldsymbol{A}^2\boldsymbol{B}) = a_1(a_1b_1 + a_2) + a_2b_1 \ .$$

The most complicated region in the TAR(2:1) process is C_4 in Condition II and we have found some strange shapes as the non-explosive regions of the solutions depending upon the parameter values of $\boldsymbol{A}$ and $\boldsymbol{B}$. Although we can give only the partial proof for our conditions due to the technical reason we shall mention in Section 5, we conjecture that the conditions are necessary and sufficient for the geometrical ergodicity in all cases. As an immediate corollary of the above two propositions, we can obtain the result originally derived by Petruccelli and Woolford (1984) for the TAR(1:1) process.

Corollary 1 : *For the TAR(1:1) model, the necessary and sufficient conditions for the geometrical ergodicity of* $\{y_t\}$ *are given by*

$$a_1 < 1 \ , b_1 < 1 \ , a_1b_1 < 1 \ . \tag{3.5}$$

3.2 TAR(2:2)

Next, we consider the TAR(2:2) process when $b_1 \leq 0$ and $b_2 = 0$. Contrary to the TAR(2:1) process, this is simpler than the case when $b_2 > 0$ in the TAR(2:2) process. In the present case we can give the necessary and sufficient conditions.

Condition III :
C_1 : $a_1 + a_2 < 1 \ , \rho(\boldsymbol{AB}^2) < 1$,
C_2 : $\rho(\boldsymbol{AB}) < 1$,
C_3 : $\rho(\boldsymbol{AB}) < 1 \ , \rho(\boldsymbol{AB}^2) < 1$,
C_4 : [*either* $a_1 + a_2 < 1$ $(0 \leq a_1 < 2, D(\boldsymbol{A}) \geq 0)$ *or* $D(\boldsymbol{A}) < 0$] , $\rho(\boldsymbol{AB}^2) < 1$.

Proposition 3 : *For the TAR(2:2) process with* $b_1 \leq 0$ *and* $b_2 = 0$, *the necessary and sufficient conditions for the geometrical ergodicity of* $\{y_t\}$ *are given by* ***Condition III***.

Second, we consider the TAR(2:2) process when $b_1 > 0$ and $b_2 = 0$. It is far more complicated than in the previous case as for the TAR(2:1) process when $b_1 \leq 0$ and $b_2 = 0$. In this case we give a set of conditions for the geometrical ergodicity and the non-explosiveness of the solution.

Condition VI :
C_1 : $a_1 + a_2 < 1 \ , 0 < b_1 < 1$,
C_2 : $a_1 + a_2 < 1$, [*there exists a non − negative* $k (\geq 2)$ *such that* $\rho\ (\boldsymbol{A}^{k-1}\ \boldsymbol{B}) < 0$
$0 \leq \rho(\boldsymbol{A}^k\boldsymbol{B}) < 1$ *and* $\rho(\boldsymbol{A}^j\boldsymbol{B}) > 1$ *for* $1 \leq j < k-1$], *and* $0 < b_1 < 1$,
C_3 : $\rho(\boldsymbol{A}^2\boldsymbol{B}) < 1, 0 < b_1 < 1$,
C_4 : [*either* $a_1 + a_2 < 1$ $(0 \leq a_1 < 2, D(\boldsymbol{A}) \geq 0)$ *or* $D(\boldsymbol{A}) < 0$] , $0 < b_1 < 1$.

Proposition 4 : *For the TAR(2:2) process with $b_1 > 0$ and $b_2 = 0$, (i) the necessary and sufficient conditions for the geometrical ergodicity of $\{y_t\}$ are given by* **Condition VI** *with C_1, C_3, and C_4, and (ii) the sufficient conditions for the near geometrical ergodicity of the solution $\{y_t\}$ are given by* **Condition VI** *with C_2.*

For an illustration, we present two figures of the ergodic and the non-explosive regions for the TAR(2:2) processes in Figure 3 and Figure 4, which are based on the simulations. Some of the conditions above can be written more explicitly by using

$$\rho(\boldsymbol{A}\boldsymbol{B}^2) = b_1(a_1 b_1 + a_2) \ .$$

We note that some of the conditions for the TAR(2:2) process in the leading case when $D(\boldsymbol{A}) < 0$ have been partially discussed by Tong (1990) without the disturbance terms. From Page 70 of Tong (1990) we set $b_1 = -0.9, a_1 = 1.8, a_2 = -0.9$ as Example 1 and $b_1 = -1.1, a_1 = 0.6, a_2 = -0.1$ as Example 2. By using Proposition 3, we can conclude that Example 1 is not geometrically ergodic while Example 2 is geometrically ergodic.

Contrary to the ergodic regions for the linear AR(2) processes, some of them are unbounded as we see in these figures. The most complicated region in the TAR(2:2) process is C_2 in Condition VI although it may be difficult to judge this finding directly from Figure 4. We have found some strange shapes as the non-explosiveness regions in C_2 depending upon the parameter values of $\boldsymbol{A}$ and $\boldsymbol{B}$. It seems that the complications involved in this case is different from the corresponding one in the region C_4 for the TAR(2:1) process when $b_1 \leq 0$. However, we have not succeeded in the more detailed characterizations of this situation except the present conditions we have obtained. Although we can give only the partial proof for our conditions due to the technical reason we shall mention to in Section 5, we conjecture that they are necessary and sufficient for the geometrical ergodicity in all cases.

As an immediate corollary of the above two propositions, we have the result obtained by Chen and Tsay (1991) for the first-order threshold model TAR(1:d) when $d = 2$.

Corollary 2 : *For the TAR(1:2) process, the necessary and sufficient conditions for the geometrical ergodicity of $\{y_t\}$ are given by*

(3.6) $$a_1 < 1 \ , b_1 < 1 \ , a_1 b_1 < 1, a_1^2 b_1 < 1, a_1 b_1^2 < 1 \ .$$

We should mention that Chen and Tsay (1991) have given the necessary and sufficient conditions for the TAR(1:d) processes with any positive integer-valued parameter *d*.

4. Some Extensions and Remarks

In this section we consider some conditions for the geometical ergodicity of the SSAR(2) processes and the general TAR(2) processes. Also we shall give some remarks on our results for the related statistical problems.

4.1 SSAR(2)

We shall utilize the Markovian representation of the SSAR(2) process, which is similar to (3.1) for the TAR(2) processes. Let $\boldsymbol{y}_t = (y_t, y_{t-1})'$ be a 2×1 vector of time series. Then the SSAR(2)process we consider is represented by

$$\boldsymbol{y}_t = \begin{cases} \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{D}\sigma_1 v_t & \text{if} \quad \boldsymbol{e}'\boldsymbol{y}_t \geq 0 \\ \boldsymbol{B}\boldsymbol{y}_{t-1} + \boldsymbol{D}\sigma_2 v_t & \text{if} \quad \boldsymbol{e}'\boldsymbol{y}_t < 0 \end{cases}, \tag{4.1}$$

where $\boldsymbol{e} = (1, -1)'$, $\boldsymbol{D} = (1, 0)'$, and the coefficient matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in this representation are given by (3.2).

We give the following conditions :

Condition V :

$$a_1 + a_2 < 1 \, , b_1 + b_2 < 1 \, , (a_1 - a_2)(b_1 - b_2) < 1 \, , \min\{|a_2| \, , |b_2|\} < 1 \, . \tag{4.2}$$

By using the coherency conditions (1.4) on a_i, b_i $(i = 1, 2)$, the above conditions can be rewritten in the parameter space of σ_i $(i = 1, 2)$ and r_i $(i = 1, 2)$ as

Condition V' :

$$r_1 > r_2 \, , r_1 + r_2 < \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \, , |r_2| < \min\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\} \, . \tag{4.3}$$

We expect that these conditions are sufficient for the geometrical ergodicity of $\{y_t\}$ in the SSAR(2) process. They are slightly weaker than the sufficient conditions

$$\rho = \min\{|a_1| + |a_2|, |b_1| + |b_2|\} < 1 \, . \tag{4.4}$$

This type of conditions for the TAR(p:1) processes ($p \geq 1$) has been obtained by Chan and Tong (1985). However, they are not necessary even for the TAR(2:1) processes as we have shown in Section 3. As an illustration, we present one figure of the geometrically ergodic regions for the SSAR(2) process in Figure 5, which is based on the simulations and drawn in the (r_1, r_2) phase. The geometrically ergodic regions in this case seem to be different from the corresponding ones in the TAR(2) processes and the ergodic region is bounded in the $(r_1, r_2)-$space due to the coherency conditions given by (1.4). The conditions for the geometrically ergodicity are considerably complicated than the results for the SSAR(1) process which can be summarized as the next proposition.

Proposition 5 : *For the SSAR(1) process, the necessary and sufficient conditions for the geometrical ergodicity of* $\{y_t\}$ *are given by*

$$a_1 < 1 \, , b_1 < 1 \, , a_1 b_1 < 1 \, . \tag{4.5}$$

These conditions can be re-written as

$$0 < r_1 < \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \, . \tag{4.6}$$

From this representation it is clear that (4.2) and (4.3) are natural generalizations of (4.5) and (4.6), respectively.

4.2 Some Concluding Remarks

In principle it would be possible to dvelop the conditions for the geometrical ergodicity and the near geometrical ergodicity for the general case of the TAR(2) processes. However, they become more complicated than in the leading case we have discussed in Section 3. In particular, it is quite tedious to write down the explicit expressions for the conditions of the geometrical ergodicity and the near geometrical ergodicity. Thus for the illustrative purpose we have simulated some figures of the geometrically ergodic regions and the non-explosive regions in the general case. Among many simulations we only present Figure 6 and Figure 7, which have complex shapes. However, they are essentially similar to those in the leading cases when the TAR(2:1) process with $b_1 \leq 0$ and the TAR(2:2) process with $b_1 > 0$.

Since the conditions for the geometrical ergodicity are the basic property of the Markovian stochastic processes, they have some implications for statistical inferences and the modelling procedure of the TAR(2) processes. In general, from our findings in this paper we expect that the higher order TAR(p) processes with an arbitrary delayed parameter d have quite complicated ergodic conditions. Although it is possible to use the least squares estimation method for the consistent estimation of the unknown parameters in the TAR processes, we need a more careful investigation on the properties of the estimation results in empirical studies. Because of our results reported in the previous sections, however, it is far beyond the scope of this paper to have a complete characterization of the stochastic processes of the TAR(p) and the SSAR(p) processes for practical usages and there still remains many statistical problems to be solved.

5. Proofs

5.1 The Method of Proofs

In this section, we give some details of the derivations and proofs of our results in Section 3. The method of proofs is basically similar to the one developed by Chen and Tsay (1991) for the TAR(1) processes. For the theoretical results on Markov chain with the general state space, see Tweedie (1975), Nummelin (1984), or Tjostheim (1990).

We shall use Lemma 1 for the Markovian representation of the TAR(2) processes of $\boldsymbol{y}_t = (y_t, y_{t-1})'$. The criterion function $g(\cdot)$ we shall use is

$$\|\boldsymbol{y}_t\| = \max[|y_t|, |y_{t-1}|] .$$

Now we prepare some notations used in this section. Let $\mathcal{F}_t$ be the $\sigma-$field generated by a sequence of random variables $\{\boldsymbol{y}_s \ , s \leq t\}$. Also define a sequence of time dependent phases for the stochastic processes : $Q_t^1 = \{y_t > 0, y_{t-1} > 0\}$, $Q_t^2 = \{y_t \leq 0, y_{t-1} > 0\}$, $Q_t^3 = \{y_t \leq 0, y_{t-1} \leq 0\}$, and $Q_t^4 = \{y_t > 0, y_{t-1} \leq 0\}$.

By using the indicator function $I(\cdot)$, we can decompose 1 into the indicator functions with four different phases as

$$1 = I(Q_t^1) + I(Q_t^2) + I(Q_t^3) + I(Q_t^4) .$$

Then we can further decompose $I(Q_t^1)$ into

$$I(Q_t^1) = I(Q_t^1 Q_{t-1}^1) + I(Q_t^1 Q_{t-1}^2) + I(Q_t^1 Q_{t-1}^2) + I(Q_t^1 Q_{t-1}^4) ,$$

for instance.

The most important technical finding in our derivations and proofs of our resutls lies in the fact that we can ignore many terms when we evaluate the growth condition (2.3) for each case.

5.2 TAR(2:1) when $b_1 > 0$

[1] First, we consider the TAR(2:1) process when $b_1 > 0$. We shall give a relatively detailed proof of our result for this particular case because the arguments in other cases have many similarities.

In this stochastic process we notice that for Q_{t+h}^4 $(h \geq 2)$ we have $y_{t+h-1} \leq 0$ and $y_{t+h} = b_1 y_{t+h-1} + v_{t+h} > 0$. It implies that $E_{t+h-1}[|y_{t+h}| I(Q_{t+h}^4)]$ and $|y_{t+h-1}|$ are bounded because we have $b_1 > 0$, $v_{t+h} > -b_1 y_{t+h-1} \geq 0$, and $2|v_{t+h}| > |y_{t+h}|$. Here we have used the notation for the conditional expectation $E_t[\,\cdot\,] = E[\,\cdot\,|\mathcal{F}_t]$ and the relation

$$\frac{1}{b_1} E_{t+h-1}[|v_{t+h}|] > |y_{t+h-1}| \geq 0 .$$

We notice that the boundedness of the conditional expectations of y_{t+h-1} and y_{t+h} imply the boundedness of the conditional expectation of y_{t+h+1}. Then sequentially we can show that $E_t[|y_{t+h+k}| I(Q_{t+h}^4)]$ are bounded for any integer $k \geq 1$. Hence we can find a positive constant $c_{1.1}$ such that for any (positive) integers $h \geq 1$ and $k \geq 1$

$$E_t[\|\boldsymbol{y}_{t+h+k}\| I(Q_{t+h}^4)] \leq c_{1.1} .$$

By using this boundedness relation, we have several consequences. For instance, since $I(Q_{t+h}^1) = I(Q_{t+h}^1 Q_{t+h-1}^1) + I(Q_{t+h}^1 Q_{t+h-1}^4)$ and $I(Q_{t+h}^2) = I(Q_{t+h}^2 Q_{t+h-1}^1) + I(Q_{t+h}^2 Q_{t+h-1}^4)$, the conditional expectations of $E_t[\|\boldsymbol{y}_{t+h}\| I(Q_{t+h}^1 Q_{t+h-1}^4)]$ and $E_t[\|\boldsymbol{y}_{t+h}\| I(Q_{t+h}^2 Q_{t+h-1}^4)]$ are bounded for any integer $h \geq 2$. Hence in the present case we only need to evaluate the conditional expectation terms associated with the four phases on the process: $I(Q_{t+h}^1 Q_{t+h-1}^1)$, $I(Q_{t+h}^2 Q_{t+h-1}^1)$, $I(Q_{t+h}^3 Q_{t+h-1}^2 Q_{t+h-2}^1)$, and $I(Q_{t+h}^3 Q_{t+h-1}^3)$.

Then we shall consider the ergodic conditions for four regions of the parameter values C_1, C_2, C_3, and C_4, separately.

[2] C_1 : In this case we first notice that $0 \leq \boldsymbol{e}_1' \boldsymbol{A} \boldsymbol{l} < 1$ and $0 \leq \boldsymbol{e}_1' \boldsymbol{A}^2 \boldsymbol{l} < 1$, where $\boldsymbol{l}' = (1,1)$. Then there exists a positive $c_{1.2}$ such that

$$\begin{aligned} (5.1) \qquad & E_t[\|\boldsymbol{y}_{t+2}\| (I(Q_{t+2}^1 Q_{t+1}^1) + I(Q_{t+2}^2 Q_{t+1}^1))] \\ \leq \;& E_t[(\boldsymbol{e}_1' \boldsymbol{A}^2 \boldsymbol{l} I_{t+2}^{(1)} + \boldsymbol{e}_1' \boldsymbol{A} \boldsymbol{l} I_{t+2}^{(2)}) \|\boldsymbol{y}_t\| (I(Q_{t+2}^1 Q_{t+1}^1) + I(Q_{t+2}^2 Q_{t+1}^1))] + c_{1.2} , \end{aligned}$$

where the indicator functions $I_{t+2}^{(1)} = I(y_{t+2} \geq y_{t+1})$ and $I_{t+2}^{(2)} = I(y_{t+2} < y_{t+1})$. Then we can take positive constants $c_{1.3}$ and $\delta_{1.1}$ $(0 \leq \delta_{1.1} < 1)$ such that

$$
\begin{aligned}
(5.2) \qquad & E_t[\|\boldsymbol{y}_{t+2}\|(I(Q_{t+2}^1 Q_{t+1}^1) + I(Q_{t+2}^2 Q_{t+1}^1))] \\
\leq \; & \delta_{1.1}\|\boldsymbol{y}_t\| E_t[I(Q_{t+2}^1 Q_{t+1}^1) + I(Q_{t+2}^2 Q_{t+1}^1)] + c_{1.3} \; .
\end{aligned}
$$

For the phase $Q_{t+2}^3 Q_{t+1}^2$, we substitute (3.1) for $\|\boldsymbol{y}_{t+2}\| = -y_{t+2} I_{t+2}^{(1)} - y_{t+1} I_{t+2}^{(2)}$ and we take positive constants $c_{1.4}$ and $\delta_{1.2}$ $(0 \leq \delta_{1.2} < 1)$ such that

$$
\begin{aligned}
(5.3) \quad & E_t[\|\boldsymbol{y}_{t+2}\| I(Q_{t+2}^3 Q_{t+1}^2)] \\
= \; & E_t[(b_1|\boldsymbol{e}_1' \boldsymbol{A}\boldsymbol{y}_t| I_{t+2}^{(1)} + |\boldsymbol{e}_1' \boldsymbol{A}\boldsymbol{y}_t| I_{t+2}^{(2)} + |v_{t+2} + b_1 v_{t+1}| I_{t+2}^{(1)} + |v_{t+1}| I_{t+1}^{(2)}) I(Q_{t+2}^3 Q_{t+1}^2)] \\
\leq \; & \delta_{1.2}\|\boldsymbol{y}_t\| E_t[I(Q_{t+2}^3 Q_{t+1}^2)] + c_{1.4} \; ,
\end{aligned}
$$

where we have taken $\delta_{1.2} = \max\{b_1(a_1 + a_2), a_1 + a_2\}$.
Similarly, for the phase $Q_{t+2}^3 Q_{t+1}^3$, we can find a positive constant $c_{1.5}$ such that

$$
(5.4) \qquad E_t[\|\boldsymbol{y}_{t+2}\| I(Q_{t+2}^3 Q_{t+1}^3)] \leq \max\{b_1^2 \, , b_1\}(-y_t) E_t[I(Q_{t+2}^3 Q_{t+1}^3)] + c_{1.5} \; .
$$

By summarizing the above inequalities on four phases, we can find positive constants $c_{1.6}$ and $\delta_{1.3}$ $(0 \leq \delta_{1.3} < 1)$ such that

$$
(5.5) \qquad E_t[\|\boldsymbol{y}_{t+2}\|] \leq \delta_{1.3}\|\boldsymbol{y}_t\| + c_{1.6} \; ,
$$

which leads to (2.3) in Lemma 1. This is because we can find a sufficiently large M and a compact set $C(M)$ depending on M such that $C = (C(M) =)\{\|\boldsymbol{y}\| \leq M\}$ and the growth condition (2.3) in Lemma 1 can be satisfied. In the following derivations we need to use this type of arguments repeatedly in each case. Because the arguments are quite similar, however, we shall not repeat them.

[3] C_2 : By repeating the procedure we have used in [2] and taking the conditional expectations, there exists a positive integer h (≥ 2) and a positive constant $c_{1.7}$ such that

$$
\begin{aligned}
(5.6) \quad E_t[\|\boldsymbol{y}_{t+h}\|] \; \leq \; & E_t\Big[\|\boldsymbol{y}_{t+h}\| \Big(I(Q_{t+h}^1 Q_{t+h-1}^1 Q_{t+h-2}^1) + I(Q_{t+h}^2 Q_{t+h-1}^1 Q_{t+h-2}^1) \\
& + I(Q_{t+h}^3 Q_{t+h-1}^2 Q_{t+h-2}^1) + I(Q_{t+h}^3 Q_{t+h-1}^3 Q_{t+h-2}^3)\Big)\Big] + c_{1.7} \; .
\end{aligned}
$$

Since $c_{1.7}$ is bounded, we need to evaluate the first four terms of (5.6). Let λ_i $(i = 1, 2)$ be the characteristic roots of $g_A(\lambda) = 0$ in (3.4). Because $a_1 < 0$ and $a_2 \geq 0$ in C_2, we have the relation that $\lambda_1 > 0 > \lambda_2$ and $|\lambda_2| > |\lambda_1|$. Then there exists a positive integer h_1 such that $\boldsymbol{e}_1' \boldsymbol{A}^{h_1} \boldsymbol{y}_t < 0$ given $\boldsymbol{y}_t \in Q_t^1$. This is because each component of $\boldsymbol{e}_1' \boldsymbol{A}^{h_1}$ eventually negative for a sufficiently large h_1. Here we write

$$
(5.7) \qquad y_{t+h_1} = \boldsymbol{e}_1' \boldsymbol{A}^{h_1} \boldsymbol{y}_t + (\sum_{i=1}^{h_1} \boldsymbol{e}_1' \boldsymbol{A}^{i-1} \boldsymbol{e}_1 v_{t+i}) \; .
$$

If we denote the second term of (5.7) as w_{t,h_1}, then we have the condition : $w_{t,h_1} > -\boldsymbol{e}_1' \boldsymbol{A}^{h_1} \boldsymbol{y}_t \geq 0$. Then there exists a positive constant $c_{1.8}$ such that

$$
E_t[\|\boldsymbol{y}_{t+h_1}\| I(\Pi_{i=0}^{h_1} Q_{t+i}^1)] \leq c_{1.8} \; .
$$

For the last term, we can use the relation that $y_{t+h_2} = b_1 y_{t+h_2-1} + v_{t+h_2}$ when $\boldsymbol{y}_{t+h_2} \in Q^3_{t+h_2}$ for a positive integer h_2 and we can take a positive $c_{1.9}$ such that

$$(5.8)\, E_t[\|\boldsymbol{y}_{t+h_2}\| I(\Pi_{i=0}^{h} Q^3_{t+h_2-i})] \le E_t[(b_1^{h_2} I_t^{(1)} + b_1^{h_2-1} I_t^{(2)})(-y_t) I(\Pi_{i=0}^{h_2} Q^3_{t+h_2-i})] + c_{1.9}.$$

We repeat the substitution of each phases on the right hand side of (5.6). If we take a sufficiently large h ($\ge h_1 + h_2 + 2$), we can reduce the second and the third terms to the first and last terms. By using the condition $0 \le b_1 < 1$ we can find positive constants $c_{1.10}$ and $\delta_{1.4}$ ($0 \le \delta_{1.4} < 1$) such that

$$(5.9) \qquad E_t[\|\boldsymbol{y}_{t+h}\|] \le \delta_{1.4} \|\boldsymbol{y}_t\| + c_{1.10}\ .$$

[4] C_3 : When $\boldsymbol{y}_{t+3} \in Q^1_{t+3}$ and $\boldsymbol{y}_{t+2} \in Q^1_{t+2}$, the equation $y_{t+3} = a_1 y_{t+2} + a_2 y_{t+1} + v_{t+3}$ implies that $v_{t+3} > (-a_1) y_{t+2} + (-a_2) y_{t+1} \ge 0$. Then by applying the same argument to $\boldsymbol{y}_{t+1}$, we have that $E_t[\|\boldsymbol{y}_{t+3}\| I(Q^1_{t+3} Q^1_{t+2} Q^1_{t+1})]$ is bounded. By using the fact that

$$I(Q^2_{t+3} Q^1_{t+2} Q^1_{t+1}) = I(Q^2_{t+3} Q^1_{t+2} Q^1_{t+1} Q^1_t) + I(Q^2_{t+3} Q^1_{t+2} Q^1_{t+1} Q^4_t)\ ,$$

and the conditional expectations $\boldsymbol{y}_{t+2}$ and $\boldsymbol{y}_{t+1}$ are bounded for the phase $Q^1_{t+1} Q^1_t$, we find that $E_t[|y_{t+2}| I(Q^1_{t+2} Q^1_{t+1})]$ is bounded and hence $E_t[\|\boldsymbol{y}_{t+3}\| I(Q^2_{t+3} Q^1_{t+2} Q^1_{t+1})]$ is also bounded.

By using that $0 < y_{t+1} = a_1 y_t + a_2 y_{t-1} + v_{t+1} < v_{t+1}$ for the phase $Q^2_{t+3} Q^1_{t+2} Q^1_{t+1} Q^1_t$, we have that $E_t[y_{t+2}]$ is bounded. Also by using the fact that $E_t[\|\boldsymbol{y}_{t+2}\| I(Q^1_{t+2} Q^1_{t+1})]$ is bounded, we have that $E_t[\|\boldsymbol{y}_{t+3}\| I(Q^3_{t+3} Q^2_{t+2} Q^1_{t+1} Q^1_t)]$ is bounded.

For the last term, there exists a positive constant $c_{1.11}$ such that

$$(5.10) \qquad E_t[\|\boldsymbol{y}_{t+3}\| I(Q^3_{t+3} Q^3_{t+2})] \le E_t[(b_1^2 I_t^{(1)} + b_1 I_t^{(2)})(-y_t) I(Q^3_{t+3} Q^3_{t+2})] + c_{1.11}\ .$$

Hence we can find positive constants $c_{1.12}$ and $\delta_{1.5}$ ($0 \le \delta_{1.5} < 1$) such that

$$(5.11) \qquad E_t[\|\boldsymbol{y}_{t+3}\|] \le \delta_{1.5} \|\boldsymbol{y}_t\| + c_{1.12}\ .$$

[5] C_4 : We consider the terms involving the phases $Q^1_{t+h} Q^1_{t+h-1}$ and $Q^2_{t+h} Q^1_{t+h-1}$ ($h \ge 2$) . Since $a_2 < 0 \le a_1$ in C_4, we need to consider two cases depending on whether the characteristic roots of $g_A(\lambda) = 0$ in (3.4) are real or complex, separately.

When $D(\boldsymbol{A}) \ge 0$ and $0 \le a_1 + a_2 < 1$, the characteristic roots are real and their absolute values are less than one. In this case we immediately see the conditions that $0 \le \boldsymbol{e}_1' \boldsymbol{A} \boldsymbol{l} < 1$ and $\boldsymbol{e}_1' \boldsymbol{A}^2 \boldsymbol{l} < 1$. Then we have the same inequality as (5.1). When $D(\boldsymbol{A}) < 0$, on the other hand, there exists a positive integer h such that (5.8) holds. Hence we can find a positive constant $\delta_{1.6}$ ($0 \le \delta_{1.6} < 1$) and $c_{1.13}$ such that the inequality (5.9) holds instead of $\delta_{1.4}$ and $c_{1.10}$.

For the remaining term involving the phase $Q^3_{t+h} Q^3_{t+h-1}$, we can use the same argument as C_2 because of the condition $0 \le b_1 < 1$.

[6] *Necessity* : For proving the necessity of our conditions, we use the similar arguments used by Petruccelli and Woolford (1984). As an illustration, consider the case when

$a_i > 0$ $(i = 1, 2)$ and $a_1 + a_2 > 1$. Then we have two real characteristic roots λ_i $(i = 1, 2)$, which satisfy the condition $\lambda_1 > 1 > 0 > \lambda_2$. Then we have two cases depending on the relative magnitudes of two roots, that is, (i) $|\lambda_1| > |\lambda_2|$ and (ii) $|\lambda_1| < |\lambda_2|$. By defining a nonsingular 2×2 matrix

$$\boldsymbol{A} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} , \tag{5.12}$$

and tranforming the original system by 2×1 vector

$$\boldsymbol{x}_t = (x_{1t}, x_{2t})' = \boldsymbol{A}^{-1}\boldsymbol{y}_t,$$

where $\boldsymbol{U}_t = (u_{1t}, u_{2t})' = \boldsymbol{A}^{-1}(v_t, 0)'$. Then we have a new representation of the process $x_{it} = \lambda_i x_{it-1} + u_{it}$ $(i = 1, 2)$.
For the case of (i), we consider $\{x_{1t}\}$ and it is not difficult to show that its solution explodes with a positive probability by the same method used of Petruccelli and Woodford (1984). For the case of (ii), we consider $\{x_{2t}\}$ and it is also not difficult to show that its solution explodes with a positive probability.

5.3 TAR(2:1) when $b_1 \leq 0$

[1] Next, we consider the TAR(2:1) process when $b_1 \leq 0$. For the phase Q^3_{t+2} in this process we have $y_{t+1} < 0$ and $y_{t+2} = b_1 y_{t+1} + v_{t+2} < 0$. When $b_1 < 0$, by using the fact that $v_{t+2} < -b_1 y_{t+1} < 0$, we notice that $E_{t+1}[\|\boldsymbol{y}_{t+2}\| I(Q^3_{t+2})]$ is bounded. When $b_1 = 0$, we need a separate consideration and decompose

$$I(Q^3_{t+2}) = I(Q^3_{t+2}Q^3_{t+1}) + I(Q^3_{t+2}Q^4_{t+1}Q^2_t) + I(Q^3_{t+2}Q^4_{t+1}Q^3_t) .$$

Then the term associated with the first phase is immediately bounded, the terms with other phases can be shown to be bounded in [2]-[5] below, and hence $E_t[|y_{t+1}| I(Q^3_{t+2})]$ is bounded. Thus $E_{t+1}[\|\boldsymbol{y}_{t+2}\| I(Q^3_{t+2})]$ is bounded when $b_1 \leq 0$, and consequently we can take a positive constant $c_{2.1}$ such that

$$E_t[\|\boldsymbol{y}_{t+h+k}\| I(Q^3_{t+h})] \leq c_{2.1}$$

for $k \geq 1$ and $h \geq 1$.
Also by decomposing

$$I(Q^4_{t+h}) = I(Q^4_{t+h}Q^2_{t+h-1}) + I(Q^4_{t+h}Q^3_{t+h-1}) ,$$

we have that $E_t[|y_{t+h}| I(Q^4_{t+h}Q^3_{t+h-1})]$ and $E_t[|y_{t+h-1}| I(Q^4_{t+h}Q^3_{t+h-1})]$ are bounded. Thus for this stochastic process the remaining phases we need to consider are $I(Q^4_{t+h}Q^2_{t+h-1})$, $I(Q^1_{t+h}) = I(Q^1_{t+h}Q^1_{t+h-1}) + I(Q^1_{t+h}Q^4_{t+h-1})$, $I(Q^2_{t+h}) = I(Q^2_{t+h}Q^1_{t+h-1}) + I(Q^2_{t+h}Q^4_{t+h-1})$. As we have done for the TAR(2:1) process when $b_1 > 0$, we shall check the ergodic conditions for each phase of the parameter space C_i $(i = 1, \cdots, 4)$ separately.

[2] C_1 : Because $a_i \geq 0$ $(i = 1, 2)$ and $a_1 + a_2 < 1$, we can find positive constants $c_{2.2}$ and $\delta_{2.1}$ $(0 \leq \delta_{2.1} < 1)$ such that for Q^1_{t+h}

$$E_t[|y_{t+h}| I(Q^1_{t+h}Q^1_{t+h-1})] \leq \delta_{2.1}|y_t| E_t[I(Q^1_{t+h}Q^1_{t+h-1})] + c_{2.2} . \tag{5.13}$$

For the phase $Q^1_{t+h}Q^4_{t+h-1}$, we have $y_{t+h-1} = b_1 y_{t+h-2} + v_{t+h-1} > 0$ and $y_{t+h-2} < 0$. Then by using a similar argument as before $E_t[|y_{t+h-1}|I(Q^1_{t+h}Q^4_{t+h-1})]$ is bounded. Thus there exist positive constants $c_{2.3}$ and $\delta_{2.2}$ $(0 \le \delta_{2.2} < 1)$ such that

$$(5.14) \qquad E_t[\|\boldsymbol{y}_{t+h}\| I(Q^1_{t+h})] \le \delta_{2.2}\|\boldsymbol{y}_t\| E_t[I(Q^1_{t+h})] + c_{2.3} \, .$$

Next, we utilize the decomposition

$$I(Q^4_{t+h}Q^2_{t+h-1}) = I(Q^4_{t+h}Q^2_{t+h-1}Q^4_{t+h-2}) + I(Q^4_{t+h}Q^2_{t+h-1}Q^1_{t+h-2}) \, .$$

When $y_{t+h-2} > 0, y_{t+h-3} > 0$, and $y_{t+h-1} = a_1 y_{t+h-2} + a_2 y_{t+h-3} + v_{t+h-1} < 0$, we have the condition that $v_{t+h-1} < -(a_1 y_{t+h-2} + a_2 y_{t+h-3}) < 0$ and $|y_{t+h-1}| \le 2|v_{t+h-1}|$. By using the same argument as before on y_{t+h}, consequently, $E_t[\|\boldsymbol{y}_{t+h}\| I(Q^4_{t+h}Q^2_{t+h-1}Q^1_{t+h-2})]$ is bounded.

The remaining phase which we need to consider is $I(\Pi^h_{i=0}(Q^4_{t+h-2i}Q^2_{t+h-1-2i}))$. By the repeated substitution of $y_{t+h} = b_1 y_{t+h-1} + v_{t+h}$, for instance, we have

$$(5.15) \qquad \begin{aligned} 0 < y_{t+h} &= b_1[a_1 y_{t+h-2} + a_2 y_{t+h-3} + v_{t+h-1}] + v_{t+h} \\ &\le (-b_1 a_2^2)(-y_{t+h-5}) + [|v_{t+h}| + b_1|v_{t+h-1}| + |b_1 a_2 v_{t+h-3}|] \, . \end{aligned}$$

By using the same substitution, we have the corresponding inequality for y_{t+h-1} . Because $a_i \ge 0$ $(i = 1, 2)$ and $a_1 + a_2 < 1$, we can take a positive integer h $(h \ge 2)$ such that $0 \le (-b_1)a_2^h < 1$ and $0 \le a_2^h < 1$. Then we can find positive constants $c_{2.4}$ and $\delta_{2.3}$ $(0 \le \delta_{2.3} < 1)$ such that

$$(5.16) \qquad \begin{aligned} & E_{t-h}[\|\boldsymbol{y}_{t+h}\| I(\prod_{i=0}^{h}(Q^4_{t+h-2i}Q^2_{t+h-1-2i}))] \\ \le\ & \delta_{2.3}|y_{t-h}| E_t[I(\prod_{i=0}^{h}(Q^4_{t+h-2i}Q^2_{t+h-1-2i}))] + c_{2.4} \, . \end{aligned}$$

For the phase $Q^2_{t+h}Q^1_{t+h-1}$, we immediately obtain that $E_t[|y_{t+h}|I(Q^2_{t+h}Q^1_{t+h-1})]$ is bounded. Also by using the same argument to $E_t[|y_{t+h-1}|I(Q^2_{t+h}Q^1_{t+h-1})]$ as before, we can find positive constants $c_{2.5}$ and $\delta_{2.4}$ $(0 \le \delta_{2.4} < 1)$ such that

$$(5.17) \qquad \begin{aligned} & E_t[|y_{t+h}|I(Q^2_{t+h}Q^1_{t+h-1})] \\ =\ & E_t[|y_{t+h}|(I(Q^2_{t+h}Q^1_{t+h-1}Q^1_t) + I(Q^2_{t+h}Q^1_{t+h-1}Q^4_{t+h-2}))] \\ \le\ & \delta_{2.4}|y_t| E_t[I(Q^2_{t+h}Q^1_{t+h-1}Q^1_{t+h-2}) + I(Q^2_{t+h}Q^1_{t+h-1}Q^4_{t+h-2})] + c_{2.5} \, . \end{aligned}$$

For the phase $Q^2_{t+h}Q^4_{t+h-1}$, we need to consider the phases $I(\Pi^h_{i=0}(Q^2_{t+h-2i}Q^4_{t+h-1-2i}))$ except several conditional expectations terms being finite. But then we can modify our arguments for the phases

$$I(\Pi^h_{i=0}(Q^4_{t+h-2i}Q^2_{t+h-1-2i}))$$

and we can find positive constants $c_{2.6}$ and $\delta_{2.5}$ $(0 \le \delta_{2.5} < 1)$ such that

$$(5.18) \qquad \begin{aligned} & E_{t-h}[\|\boldsymbol{y}_{t+h}\| I(\prod_{i=0}^{h}(Q^2_{t+h-2i}Q^4_{t+h-1-2i}))] \\ \le\ & \delta_{2.5}(-y_{t-h}) E_{t-h}[I(\prod_{i=0}^{h}(Q^2_{t+h-2i}Q^4_{t+h-1-2i}))] + c_{2.6} \, . \end{aligned}$$

Hence together with (5.15)-(5.18) we can find positive constants $c_{2.7}$, $\delta_{2.6}$ ($0 \leq \delta_{2.6} < 1$), and a common positive integer h such that

$$E_t[\|\boldsymbol{y}_{t+h}\|] \leq \delta_{2.6}\|\boldsymbol{y}_t\| + c_{2.7} \,. \tag{5.19}$$

[3] C_2 : First, we use the decomposition

$$I(Q_{t+h}^4 Q_{t+h-1}^2) = I(Q_{t+h}^4 Q_{t+h-1}^2 Q_{t+h-2}^4) + I(Q_{t+h}^4 Q_{t+h-1}^2 Q_{t+h-2}^1) \,.$$

For the first phase, we have $y_{t+h-1} = a_1 y_{t+h-2} + a_2 y_{t+h-3} + v_{t+h-1}$ and $y_{t+h} = b_1 y_{t+h-1} + v_{t+h}$. Then there exists a constant $c_{2.8}$ such that

$$\begin{aligned} & E_{t+h-3}[\|\boldsymbol{y}_{t+h}\| I(Q_{t+h}^4 Q_{t+h-1}^2 Q_{t+h-2}^4)] \\ \leq \; & \max[a_1 b_1 + a_2, (-b_1)(a_1 b_1 + a_2)](-y_{t+h-3}) I(Q_{t+h}^4 Q_{t+h-1}^2 Q_{t+h-2}^4)] + c_{2.8} \,. \end{aligned} \tag{5.20}$$

By repeating this procedure, we can find a positive integer h (≥ 2) such that $|(-b_1)(a_1 b_1 + a_2)^h| < 1$ because of the condition $0 \leq a_1 b_1 + a_2 < 1$. Hence we can take positive constants $c_{2.9}$ and $\delta_{2.7}$ ($0 \leq \delta_{2.7} < 1$) such that

$$\begin{aligned} & E_{t-h}[\|\boldsymbol{y}_{t+h}\| I(\prod_{i=0}^{h}(Q_{t+h-2i}^4 Q_{t+h-1-2i}^2))] \\ \leq \; & \delta_{2.7}|y_{t-h}| E_{t-h}[I(\prod_{i=0}^{h}(Q_{t+h-2i}^4 Q_{t+h-1-2i}^2))] + c_{2.9} \,. \end{aligned} \tag{5.21}$$

This type of argument can be also applicable to the terms involving $\Pi_{i=0}^{h}(Q_{t+h-2i}^2 Q_{t+h-1-2i}^4)$. Since

$$I(Q_{t+h}^2 Q_{t+h-1}^4) = I(Q_{t+h}^2 Q_{t+h-1}^4 Q_{t+h-2}^2) + I(Q_{t+h}^2 Q_{t+h-1}^4 Q_{t+h-2}^3)$$

and $E_t[|y_{t+h}| I(Q_{t+h}^2 Q_{t+h-1}^4 Q_{t+h-2}^3)]$ is bounded, we can also find positive constants $c_{2.10}$ and $\delta_{2.8}$ ($0 \leq \delta_{2.8} < 1$) such that

$$\begin{aligned} & E_{t-h}[\|\boldsymbol{y}_{t+h}\| I(\prod_{i=0}^{h}(Q_{t+h-2i}^2 Q_{t+h-1-2i}^4))] \\ \leq \; & \delta_{2.8}|y_{t-h}| E_{t-h}[I(\prod_{i=0}^{h}(Q_{t+h-2i}^2 Q_{t+h-1-2i}^4))] + c_{2.10} \,. \end{aligned} \tag{5.22}$$

Next, we consider the phase Q_{t+h}^1 ($h \geq 2$) and use the decomposition

$$I(Q_{t+h}^1) = I(Q_{t+h}^1 Q_{t+h-1}^4) + I(Q_{t+h}^1 Q_{t+h-1}^1) \,,$$

whose first term can be further decomposed as

$$I(Q_{t+h}^1 Q_{t+h-1}^4) = I(Q_{t+h}^1 Q_{t+h-1}^4 Q_{t+h-2}^2) + I(Q_{t+h}^1 Q_{t+h-1}^4 Q_{t+h-2}^3) \,.$$

For the phase $Q_{t+h}^1 Q_{t+h-1}^4 Q_{t+h-2}^3$, the conditional expectations of all terms involving

y_{t+h-i} $(i=0,1,\cdots,h-2)$ are bounded. For the phase $Q^1_{t+h}Q^4_{t+h-1}Q^2_{t+h-2}$, the conditional expectations of all terms can be reduced to that with $I(Q^4_{t+h-1}Q^2_{t+h-2})$.

Now we consider the decomposition of

$$I(Q^1_{t+h}Q^2_{t+h-1}) = I(Q^1_{t+h}Q^1_{t+h-1}Q^1_{t+h-2}) + I(Q^1_{t+h}Q^1_{t+h-1}Q^4_{t+h-2}) \ .$$

By using the argument as in [1], for $I(Q^1_{t+h}Q^1_{t+h-1}Q^1_{t+h-2})$ we can find positive constants $c_{2.11}$ and $\delta_{2.9}$ $(0 \le \delta_{2.9} < 1)$ such that

$$\begin{aligned} (5.23) \qquad & E_{t+h-2}[\|\boldsymbol{y}_{t+h}\| I(Q^1_{t+h}Q^1_{t+h-1}Q^1_{t+h-2})] \\ & \le \ \delta_{2.9}\|y_{t+h-2}\| E_{t+h-2}[I(Q^1_{t+h}Q^1_{t+h-1}Q^1_{t+h-2})] + c_{2.11} \end{aligned}$$

simply because of the condition $0 \le a_2 < 1$.
For the phase $Q^1_{t+h}Q^1_{t+h-1}Q^4_{t+h-2}$, we can ignore the conditional expectation terms except that with $I(Q^1_{t+h}Q^1_{t+h-1}Q^4_{t+h-2}Q^3_{t+h-3})$, which is eventually reduced to the term with the phase $I(Q^4_{t+h-2}Q^3_{t+h-3})$.
Finally, we consider the phase Q^2_{t+h} . We need to evaluate

$$I(Q^2_{t+h}Q^1_{t+h-1}) = I(Q^2_{t+h}Q^1_{t+h-1}Q^1_{t+h-2}) + I(Q^2_{t+h}Q^1_{t+h-1}Q^4_{t+h-2}) \ .$$

But then we can use the same argument as for the phases $Q^1_{t+h}Q^1_{t+h-1}$ and $Q^1_{t+h}Q^4_{t+h-1}$.

[4] C_3 : Because many arguments in [4] are similar to [3], we only present the important differences. As we have considered in [3], for the phase Q^4_t we only need to investigate the decomposition

$$I(Q^4_{t+h}Q^2_{t+h-1}) = I(Q^4_{t+h}Q^2_{t+h-1}Q^4_{t+h-2}Q^2_{t+h-3}) + I(Q^4_{t+h}Q^2_{t+h-1}Q^1_{t+h-2}Q^4_{t+h-3}Q^2_{t+h-4}) + \cdots \ ,$$

where we have ignored other terms. For the first term of $I(Q^4_{t+h}Q^2_{t+h-1})$, we can use the same argument as [3]. For the second term, we can find a positive constant $c_{2.12}$ and an integer h such that

$$\begin{aligned} (5.24) \quad & E_{t+h-4}[\max[y_{t+h}, -y_{t+h-1}] I(Q^4_{t+h}Q^2_{t+h-1}Q^1_{t+h-2}Q^4_{t+h-3}Q^2_{t+h-4})] \\ \le \ & \max[-b_1\boldsymbol{e}_1'\boldsymbol{A}^2\boldsymbol{b}, \boldsymbol{e}_1'\boldsymbol{A}^2\boldsymbol{b}](-y_{t+h-4})E_{t+h-4}[I(Q^4_{t+h}Q^2_{t+h-1}Q^1_{t+h-2}Q^4_{t+h-3}Q^2_{t+h-4})] \\ & + c_{2.12}. \end{aligned}$$

Then we can find an integer h, and positive constants $c_{2.13}$ and $\delta_{2.10}$ $(0 \le \delta_{2.10} < 1)$ such that

$$(5.25) \qquad E_t[\|\boldsymbol{y}_{t+h}\| I(Q^4_{t+h})] \le \delta_{2.10}\|\boldsymbol{y}_t\| E_t[I(Q^4_{t+h})] + c_{2.13} \ ,$$

provided that $\boldsymbol{e}_1'\boldsymbol{A}\boldsymbol{b} < 1$ and $\boldsymbol{e}_1'\boldsymbol{A}^2\boldsymbol{b} < 1$.

[5] C_4 : There are some complications involved in this case and the ergodic region becomes quite complex. We have two situations whether the characteristic roots of the coefficient matrix $\boldsymbol{A}$ ($g_A(\lambda) = 0$) are real or complex.
When $D(\boldsymbol{A}) \ge 0$ $(0 \le a_1 \le 2)$ and $a_1 + a_2 < 1$, two roots λ_i $(i = 1,2)$ are real and we have a simple relation that $0 \le \lambda_2 \le \lambda_1 < 1$. Then as before in Sections 5.2 and 5.3 we can find a positive integer h such that $\boldsymbol{e}'\boldsymbol{A}^h\boldsymbol{l} < 0$. Thus we can find positive constants $c_{2.14}$ and $\delta_{2.11}$ $(0 \le \delta_{2.11} < 1)$ such that

$$(5.26) \qquad E_t[\|\boldsymbol{y}_{t+h}\|] \le \delta_{2.11}\|\boldsymbol{y}_t\| + c_{2.14} \ ,$$

When $\boldsymbol{D}(A) < 0$, there are two complex roots and the conditions become complicated. For the phase Q_t^4, we decompose

$$I(Q_{t+h}^4) = I(Q_{t+h}^4 Q_{t+h-1}^2 Q_{t+h-2}^1 Q_{t+h-3}^4) + I(Q_{t+h}^4 Q_{t+h-1}^2 Q_{t+h-2}^1 Q_{t+h-3}^1) + \cdots ,$$

where there are some other terms. But the conditional expectations with those terms are bounded by using our previous arguments, and can be ignored. However, we need to consider the transitions from phases Q_{t+h-1}^1 and Q_{t+h-1}^2 to the phase Q_{t+h}^1, which may have many periodicities. For instance, as we have given in [4], we can find an integer h_1 and a positive constant $c_{2.15}$ such that

$$\begin{aligned} (5.27)\quad & E_{t+h_1-4}[\|\boldsymbol{y}_{t+h_1}\| I(Q_{t+h_1}^4 Q_{t+h_1-1}^2 Q_{t+h_1-2}^1 Q_{t+h_1-3}^4)] \\ \leq\ & \max[-b_1, 1]\boldsymbol{e}_1' \boldsymbol{A}^2 \boldsymbol{b}(-y_{t+h_1-4}) E_{t+h_1-4}[I(Q_{t+h_1}^4 Q_{t+h_1-1}^2 Q_{t+h_1-2}^1 Q_{t+h_1-3}^4)] + c_{2.15}. \end{aligned}$$

By the same token, we can find an integer h_2 and a positive constant $c_{2.16}$ such that

$$\begin{aligned} (5.28)\quad & E_{t+h_2-5}[\|\boldsymbol{y}_{t+h_2}\| I(Q_{t+h_2}^4 Q_{t+h_2-1}^2 Q_{t+h_2-2}^1 Q_{t+h_2-3}^1 Q_{t+h_2-4}^4)] \\ \leq\ & \max[-b_1, 1]\boldsymbol{e}_1' \boldsymbol{A}^3 \boldsymbol{b}(-y_{t+h_2-5}) E_{t+h_2-5}[I(Q_{t+h_2}^4 Q_{t+h_2-1}^2 Q_{t+h_2-2}^1 Q_{t+h_2-3}^1 Q_{t+h_2-4}^4)] \\ & + c_{2.16}. \end{aligned}$$

For these cases, we can take positive integers h_1 (≥ 2) and h_2 (≥ 2) such that

$$\max[(-b_1)(\boldsymbol{e}_1' \boldsymbol{A}^2 \boldsymbol{b})^{h_1}, (\boldsymbol{e}_1' \boldsymbol{A}^2 \boldsymbol{b})^{h_1}] < 1 ,$$

and

$$\max[(-b_1)(\boldsymbol{e}_1' \boldsymbol{A}^3 \boldsymbol{b})^{h_2}, (\boldsymbol{e}_1' \boldsymbol{A}^3 \boldsymbol{b})^{h_2}] < 1 .$$

Then we can deal with other cases as well in the same ways provided that we have finite positive integers h_i $(i \geq 1)$. Hence we have proved that the solutions are non-explosive if the conditions in this case were met.

[6] *Necessity* : In order to prove the necessity of our conditions for C_1, C_2, and C_3 cases, we use the similar arguments used for the TAR(2:1) model when $b_1 > 0$, which is in turn based on the method used by Petruccelli and Woolford (1984). Since it is straightforward to do it, we omit the details.

5.4 TAR(2:2) when $b_1 \leq 0$

[1] In this subsection we shall consider the TAR(2:2) process when $b_1 \leq 0$. For this stochastic process we notice that $E_t[\|\boldsymbol{y}_t\| I(Q_t^1 Q_{t-1}^4)]$ and $E_t[\|\boldsymbol{y}_t\| I(Q_t^3 Q_{t-1}^3)]$ are bounded. Then except several (finite) conditional expectations we only need to investigate the phase $Q_t^1 Q_{t-1}^1 Q_{t-2}^1$ for the phase Q_t^1,

$$I(Q_t^2 Q_{t-1}^1 Q_{t-2}^1) + I(Q_t^2 Q_{t-1}^4)$$

for the phase Q_t^2, and

$$I(Q_t^3 Q_{t-1}^2 Q_{t-2}^1 Q_{t-3}^1) + I(Q_t^3 Q_{t-1}^2 Q_{t-2}^4)$$

for the phase Q_t^3, respectively. Also for the phase Q_t^4, we only need to investigate

$$I(Q_t^4 Q_{t-1}^3 Q_{t-2}^2) + I(Q_t^4 Q_{t-1}^2) .$$

As for the TAR(2:1) process, we need to consider the ergodic conditions for C_1, C_2, C_3, and C_4, separately.

[2] : First, we consider C_1. Because $a_i \geq 0$ $(i = 1, 2)$ and $a_1 + a_2 < 1$, we have the condition that $\boldsymbol{e}_1' \boldsymbol{A l} < 1$ and $0 \leq \boldsymbol{e}_1' \boldsymbol{A}^2 \boldsymbol{l} < 1$. Then there exist an integer h and positive constants $c_{3.1}$ and $\delta_{3.1}$ $(0 \leq \delta_{3.1} < 1)$ such that

$$\begin{aligned} (5.29) \qquad & E_{t+h-2}[\|\boldsymbol{y}_{t+h}\|(I(Q_{t+h}^1 Q_{t+h-1}^1) + I(Q_{t+h}^2 Q_{t+h-1}^1 Q_{t+h-2}^1))] \\ \leq\ & \delta_{3.1}\|\boldsymbol{y}_{t+h-2}\| E_{t+h-2}[I(Q_{t+h}^2 Q_{t+h-1}^1) + I(Q_{t+h}^2 Q_{t+h-1}^1)] + c_{3.1} . \end{aligned}$$

For the phase $Q_{t+h}^2 Q_{t+h-1}^4$, by using the similar arguments in Sections 5.2 and 5.3, we only need to investigate the conditional expectations with the phases

$$I(Q_{t+h}^2 Q_{t+h-1}^4 Q_{t+h-2}^3 Q_{t+h-3}^2 Q_{t+h-4}^4) + I(Q_{t+h}^2 Q_{t+h-1}^4 Q_{t+h-2}^2 Q_{t+h-3}^4)$$

For the second term, we have

$$\begin{aligned} (5.30) \qquad & E_{t+h_1-3}[\|\boldsymbol{y}_{t+h_1}\| I(Q_{t+h_1}^2 Q_{t+h_1-1}^4 Q_{t+h_1-2}^2 Q_{t+h_1-3}^4)] \\ \leq\ & \delta_{3.2}\|\boldsymbol{y}_{t+h_1-3}\| E_{t+h_1-3}[I(Q_{t+h_1}^2 Q_{t+h_1-1}^4 Q_{t+h_1-2}^2 Q_{t+h_1-3}^4)] + c_{3.2} , \end{aligned}$$

where h_1 is a positive integer, $c_{3.2}$ is a positive constant, and $\delta_{3.2} = \max[-b_1, 1]\boldsymbol{e}_1' \boldsymbol{A b}$. If $\boldsymbol{e}_1' \boldsymbol{A b} < 0$ and $b_1 < 0$, then we have a representation

$$0 > y_{t+h_1} = b_1(a_1 b_1 + a_2) y_{t+h_1-3} + w_{t+h_1}$$

with $y_{t+h_1-3} \in Q_{t+h_1-3}^4$ and $y_{t+h_1} \in Q_{t+h_1}^2$, where w_{t+h_1} is a linear combination of v_{t+h_1-i} $(i = 0, 1, 2)$.
Then we have that

$$E_{t+h-3}[|y_{t+h_1}| I(Q_{t+h_1}^2 Q_{t+h_1-1}^4 Q_{t+h_1-2}^2 Q_{t+h_1-3}^4)]$$

is bounded. By using this fact, we also have

$$E_{t+h-3}[|y_{t+h_1-1}| I(Q_{t+h_1}^2 Q_{t+h_1-1}^4 Q_{t+h_1-2}^2 Q_{t+h_1-3}^4)]$$

is bounded. We note that we need a separate consideration when $b_1 = 0$, but the result is the same by using a tedious but similar argument as we have used in Section 5.3.
For the first term, we have

$$\begin{aligned} (5.31) \qquad & E_{t+h_2-4}[\|\boldsymbol{y}_{t+h_2}\| I(Q_{t+h_2}^2 Q_{t+h_2-1}^4 Q_{t+h_2-2}^3 Q_{t+h_2-3}^2 Q_{t+h_2-4}^4)] \\ \leq\ & \delta_{3.3}\|\boldsymbol{y}_{t+h_2-4}\| E_{t+h_2-4}[I(Q_{t+h_2}^2 Q_{t+h_2-1}^4 Q_{t+h_2-2}^3 Q_{t+h_2-3}^2 Q_{t+h_2-4}^4)] + c_{3.3} , \end{aligned}$$

where h_2 is a positive integer, $c_{3.3}$ is a positive constant, and $\delta_{3.3} = \max[-b_1, 1]\boldsymbol{e}_1' \boldsymbol{A B}^2 \boldsymbol{e}_1$. If $\boldsymbol{e}_1' \boldsymbol{A B}^2 \boldsymbol{e}_1 < 0$, then we have the similar argument as for $\boldsymbol{e}_1' \boldsymbol{A b} < 0$ and we only need to consider the case when $e_1' AB^2 e_1 \geq 0$. Under the conditions we have stated in this case, we can find positive integers h_1 and h_2 such that

$$\max[-b_1, 1](e_1' ABe_1)^{h_1} < 1$$

and

$$\max[-b_1, 1](e_1' AB^2 e_1)^{h_2} < 1 \, .$$

Actually, because we have $a_i \geq 0$ $(i = 1, 2)$ and $a_1 + a_2 < 1$ for C_1, the first condition is automatically satisfied while we need the same condition explicitly for C_2 and C_3.
Hence we can find an integer h and positive constants $c_{3.4}$ and $\delta_{3.4}$ $(0 \leq \delta_{3.4} < 1)$ such that

$$E_t[\|\boldsymbol{y}_{t+h}\| I(Q_{t+h}^2 Q_{t+h-1}^4)] \leq \delta_{3.4} \|\boldsymbol{y}_t\| E[I(Q_{t+h}^2 Q_{t+h-1}^4)] + c_{3.4} \, . \tag{5.32}$$

The rest of our arguments are quite similar to those appeared in the previous subsections. Also it is straightforward to treat the phases for C_i $(i = 2, 3, 4)$. The proof of the necessity of our conditions is also quite similar to the one for the TAR(2:1) when $b_1 > 0$ and so we omit its details.

5.5 TAR(2:2) when $b_1 > 0$

[1] The proof for the TAR(2:2) process when $b_1 > 0$ is quite similar to the ones for the TAR(2:1) process when $b_1 \leq 0$. We notice that $E_t[\|\boldsymbol{y}_t\| I(Q_t^2 Q_{t-1}^4)]$ and $E_t[\|\boldsymbol{y}_t\| I(Q_t^4 Q_{t-1}^3)]$ are bounded. Then for the phases Q_t^2 and Q_t^4, we only need to investigate the conditional expectations with the phases $Q_t^2 Q_{t-1}^1$ and $Q_t^4 Q_{t-1}^2$.
For the phases Q_t^1 and Q_t^3, we can utilize the decomposition

$$I(Q_t^1) = I(Q_t^1 Q_{t-1}^1) + I(Q_t^1 Q_{t-1}^4)$$

and

$$I(Q_t^3) = I(Q_t^3 Q_{t-1}^3) + I(Q_t^3 Q_{t-1}^2) \, .$$

[2] : The arguments for C_i $(i = 1, 3, 4)$ are quite similar to the cases already appeared in the previous subsections and we briefly discuss only the case for C_2.
The derivations for C_2 are quite similar to the case of C_4 for the TAR(2:1) process when $b_1 \leq 0$. In the present case, by using successive substitutions, we use the decomposition

$$\begin{aligned} I(Q_t^1 Q_{t-1}^1) + I(Q_t^1 Q_{t-1}^4) &= I(Q_t^1 Q_{t-1}^1 Q_{t-2}^1 Q_{t-3}^1) + I(Q_t^1 Q_{t-1}^1 Q_{t-2}^1 Q_{t-3}^4) \\ &\quad + I(Q_t^1 Q_{t-1}^1 Q_{t-2}^4 Q_{t-3}^2) + I(Q_t^1 Q_{t-1}^4 Q_{t-2}^2 Q_{t-3}^1) + \cdots \, . \end{aligned} \tag{5.33}$$

As we have discussed in [1], we can ignore other phases because the associated conditional expectations are bounded and they can be negligible.
In the present situation we have to only take account of the sequences of the phases $[Q^1 \to Q^2 \to Q^4]$, $[Q^1 \to Q^1 \to Q^2 \to Q^4]$, and so on. For the first case, we can take a positive integer h_1 such that

$$\begin{aligned} & E_{t+h_1-4}[\|\boldsymbol{y}_{t+h_1}\| I(Q_{t+h_1}^1 Q_{t+h_1-1}^4 Q_{t+h_1-2}^2 Q_{t+h_1-3}^1)] \\ \leq \; & \delta_{4.1}(y_{t+h_1-4}) E_{t+h_1-4}[I(Q_{t+h_1}^1 Q_{t+h_1-1}^4 Q_{t+h_1-2}^2 Q_{t+h_1-3}^1)] + c_{4.1} \, , \end{aligned} \tag{5.34}$$

where we have taken $c_{4.1}$ is a positive constant.
In this case we can find a sufficiently large integer h_1 (≥ 2) such that

$$\delta_{4.1} = \max[b_1, 1](\boldsymbol{e}_1^{'}\boldsymbol{A}^2\boldsymbol{b})^{h_1} < 1 \ .$$

The rest of our arguments are quite similar to the case of C_2 for the TAR(2:1) process when $b_1 \leq 0$. We have proven that the solutions are non-explosive if the conditions in this case were met.

The proof of the necessity of our conditions for C_1, C_3, and C_4 is also quite similar to those in other cases.

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Appendix : Some Figures

In this appendix, we give several figures. All figures reported here are the results of the simulations on the sets of 10,000 realizations of $\{y_t\}$ based on the TAR(2) processes and the SSAR(2) processes without any disturbance terms. We have checked the ergodic regions and basically confirmed the adequacy of the same regions by the corresponding simulations for the TAR(2) and SSAR(2) processes with disturbances. It was all we could do because the criteria of convergence in simulations are more difficult and subtle when there are noise terms.

All figures for the TAR(2:d) processes are denoted by TAR(2) with the delayed parameter d and drawn in the (a_1, a_2) space while the figure for the SSAR(2) process are drawn in the $(r_1, r_2)-$space. The shaded areas in figures are the geometrically ergodic regions.

Appendix : Some Figures (Continued)

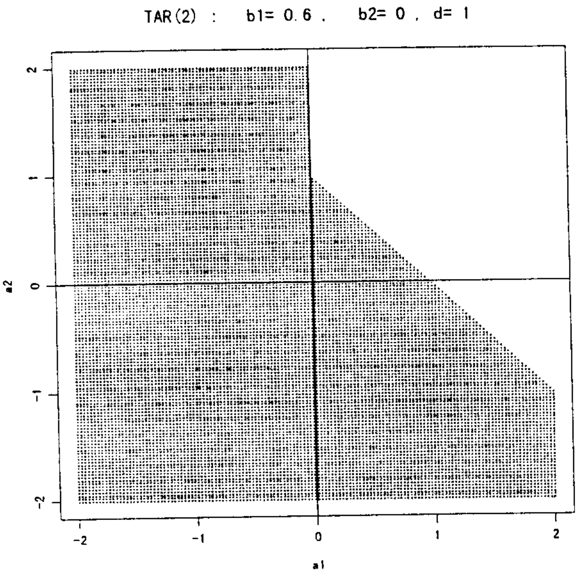

Figure 1

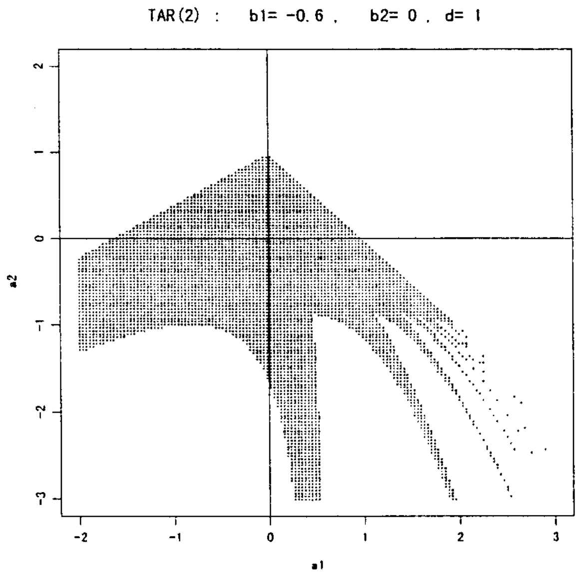

Figure 2

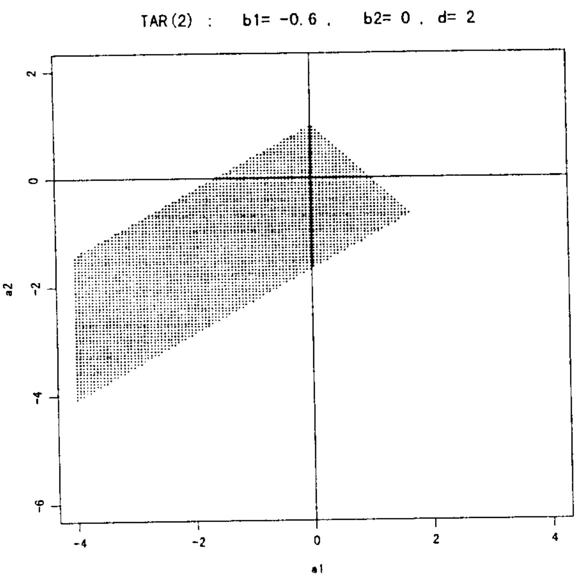

Figure 3

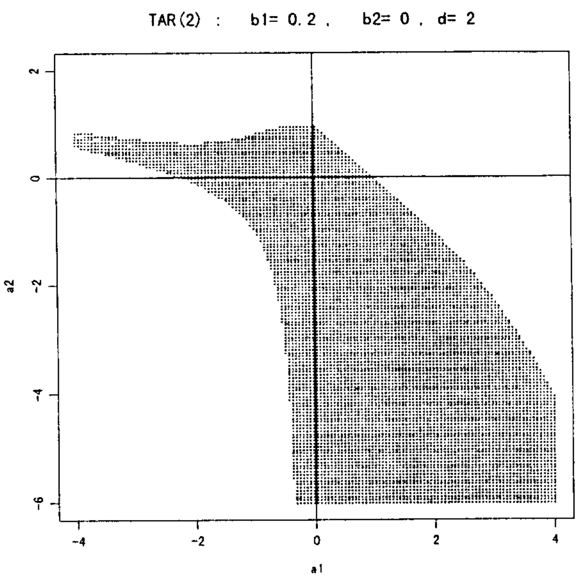

Figure 4

Appendix : Some Figures (Continued)

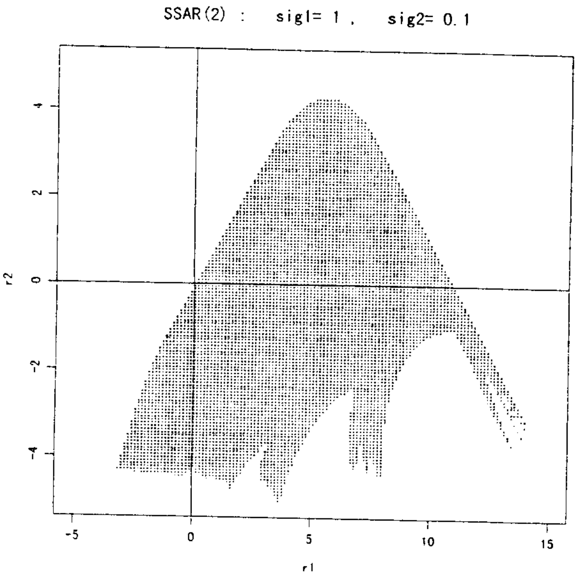

Figure 5

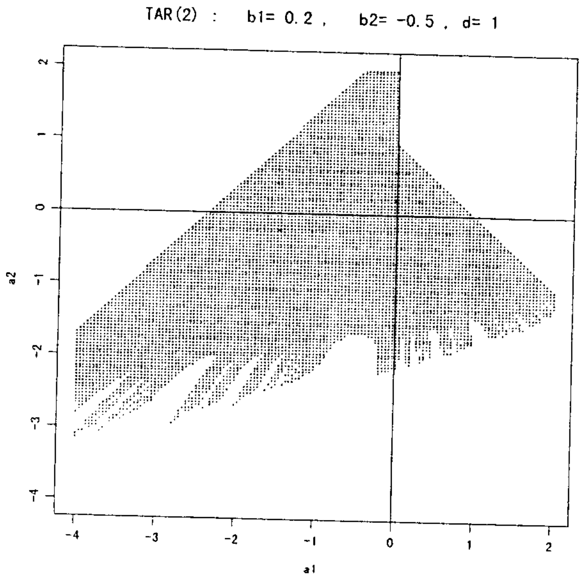

Figure 6

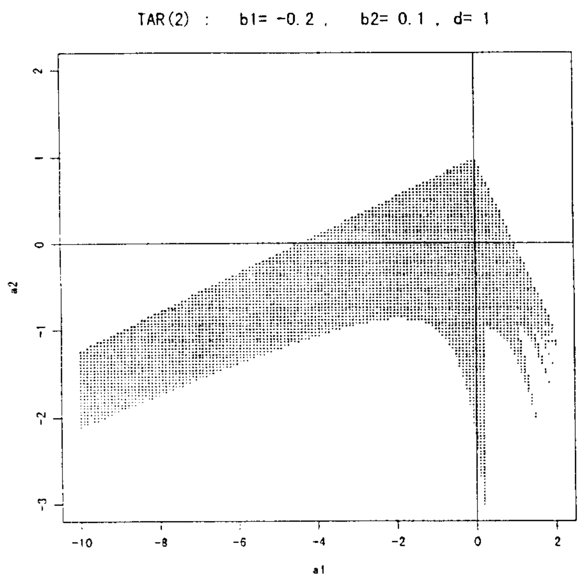

Figure 7